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Orthonormal polynomials with generalized Freud-type weights

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Abstract

We consider a certain generalized Freud-type weight $W_{rQ}^2(x) = |x|^{2r}\exp(-2Q(x))$, where $r > -\frac{1}{2}$ and $Q: \mathbf{R} \to \mathbf{R}$ is even and continuous, Q' is continuous, $Q' > 0$ in $(0, \infty)$, and Q'' is continuous in $(0, \infty)$. Furthermore, Q satisfies further conditions. Recently, Levin and Lubinsky have studied the sequence of orthonormal polynomials $\{P_n(W_Q^2; x)\}_{n=0}^{\infty}$ with the Freud weight $W_Q^2(x) = \exp(-2Q(x))$ on $\mathbf{R}$, and then they have obtained many interesting properties of $P_n(W_Q^2; x)$ [LL1]. We investigate the properties of $P_n(W_{rQ}^2; x)$, which contain extensions of Levin and Lubinsky's results and improvements of Bauldry's results [Ba1,LL1].

0. Introduction

Let $Q: \mathbf{R} \to \mathbf{R}$ be even and continuous, Q' be continuous, $Q' > 0$ in $(0, \infty)$, and let Q'' be continuous in $(0, \infty)$. Furthermore, Q satisfies the following condition:

$$1 < A \leqslant \{(d/dx)(xQ'(x))\}/Q'(x) \leqslant B, \quad x \in (0, \infty), \tag{0.1}$$

where A and B are constants. Then

$$W_Q(x) = \exp(-Q(x)) \tag{0.2}$$

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is called a Freud weight, and the typical case is

$$W_\alpha(x) = \exp(-|x|^\alpha), \quad \alpha > 1.$$

Recently, Levin and Lubinsky have studied the sequence of orthonormal polynomials $\{P_n(W_Q^2; x)\}_{n=0}^{\infty}$ with the Freud weight (0.2) on $\mathbf{R}$, and then they have obtained many interesting properties of $P_n(W_Q^2; x)$ [LL1].

In this paper we treat certain generalized Freud-type weights

$$W_{rQ}(x) = |x|^r \exp(-Q(x)) \quad (x \in \mathbf{R}, 2r > -1), \tag{0.3}$$

and study the series of orthonormal polynomials $\{P_n(W_{rQ}^2; x)\}_{n=0}^{\infty}$ with weight (0.3), where $\{P_n(W_{rQ}^2; x)\}_{n=0}^{\infty}$ are constructed by

$$\int_{-\infty}^{\infty} P_i(W_{rQ}^2; t) P_j(W_{rQ}^2; t) W_{rQ}^2(t)\, dt = \delta_{ij} \quad \text{(Kronecker's delta)},$$
$$i, j = 0, 1, 2, \dots \, . \tag{0.4}$$

We investigate the properties of $P_n(W_{rQ}^2; x)$, which contain extensions of Levin and Lubinsky's results and improvements of Bauldry's results [Ba1,LL1].

We will use the same constant C even if it is different in the same line. If for two sequences $\{c_n\}_{n=1}^{\infty}$ and $\{d_n\}_{n=1}^{\infty}$ there are two positive numbers C, D such that $C \leqslant c_n/d_n \leqslant D$, then we denote as $c_n \sim d_n$.

1. Preliminaries and theorems

First, we denote the fundamental definitions. For Q satisfying (0.1) and $2r > -1$, we define the weight $W_{rQ}(x)$ in (0.3), and construct the orthonormal polynomials $\{P_n(W_{rQ}^2; x)\}$ in (0.4), where

$$P_n(x) = P_n(W_{rQ}^2; x) = \gamma_n x^n + \cdots +, \quad \gamma_n = \gamma_n(W_{rQ}^2) > 0, \quad n = 1, 2, 3, \dots \, .$$

We define $b_n = \gamma_{n-1}/\gamma_n$, and denote the zeros of $P_n(W_{rQ}^2; x)$ by $-\infty < x_{nn} < x_{n-1,n} < \cdots < x_{2n} < x_{1n} < \infty$. Using the reproducing kernel

$$K_n(x, t) = b_n\{P_n(x)P_{n-1}(t) - P_n(t)P_{n-1}(x)\}/(x - t), \tag{1.1}$$

we define the Christoffel function $\lambda_n(W_{rQ}^2; x)$,

$$\lambda_n^{-1}(x) = \lambda_n^{-1}(W_{rQ}^2; x) = K_n(x, x) = b_n\{P_n'(x)P_{n-1}(x) - P_n(x)P_{n-1}'(x)\}, \tag{1.2}$$

where the Cotes number is given by $\lambda_{kn} = \lambda_n(x_{kn})$ (see [Ne1, (3.3),(3.6)]), and satisfies

$$\lambda_{kn}^{-1} = b_n P_n'(x_{kn}) P_{n-1}(x_{kn}). \tag{1.3}$$

The Mhaskar–Rahmanov–Saff number a_u is the unique positive root of the equation

$$u = (2/\pi) \int_0^1 a_u t Q'(a_u t)(1 - t^2)^{-1/2} dt, \quad u > 0.$$

We also consider the root $x = q_u > 0$ of $u = xQ'(x)$ for $u > 0$. Then we have

$$a_n \sim q_n \sim b_n \sim x_{1n}, \quad n = 1, 2, 3, \ldots, \ ([\text{Ba1, Theorem 3.5, LL1}]). \tag{1.4}$$

The generalized Christoffel function $\lambda_{np}(x)$ is defined by

$$\lambda_{np}(x) = \inf_{P \in \prod_{n-1}} \int_{-\infty}^{\infty} |P(t)|^p W_{rQ}^p(t)\, dt / |P(x)|^p, \quad 0 < p < \infty, \tag{1.5}$$

where $\prod_n$ denotes the class of polynomials with degree at most n. The function $\lambda_n(x)$ in (1.2) is a special case of (1.5) for $p = 2$. We will give an estimate of $\lambda_{np}(x)$.

The following theorem is an improvement of [Ba1, Theorem 3.1], where for $r = 0$ it is given by Levin and Lubinsky [LL1, Theorem 1.8] (this is called the infinite–finite range inequalities).

Theorem 1.1. *We assume $pr + 1 > 0$ if $0 < p < \infty$, and $r \geq 0$ if $p = \infty$. Let $K > 0$. Then for every $P \in \prod_n$ we have*

$$||PW_{rQ}||_{L_p(R)} \leq C||PW_{rQ}||_{L_p(|x| \leq a_n(1-Kn^{-2/3}))}.$$

We can improve a result of Bauldry's (see Lemma 2.3) for $\lambda_{n2}(x) = \lambda_n(W_{rQ}^2; x)$.

Theorem 1.2. *Let $L > 0$ and $\varepsilon > 0$.*

(i) *For $|x| < \varepsilon a_n/n$ we have $\lambda_n(W_{rQ}^2; x) \sim (a_n/n)^{2r+1}$.*

(ii) *For $\varepsilon a_n/n \leq |x|$ we have*

$$\lambda_n(W_{rQ}^2; x) \geq C(a_n/n) W_{rQ}^2(x)[\max\{n^{-2/3}, 1 - (|x|/a_n)\}]^{-1/2}.$$

(iii) *For $\varepsilon a_n/n \leq |x| \leq a_n(1 + Ln^{-2/3})$, we see*

$$\lambda_n(W_{rQ}^2; x) \sim (a_n/n) W_{rQ}^2(x)[\max\{n^{-2/3}, 1 - (|x|/a_n)\}]^{-1/2}.$$

The maximum zero x_{1n} of $P_n(W_{rQ}^2; x)$ is estimated as follows.

Theorem 1.3. *There is a certain constant C such that*

$$|(x_{1n}/a_n) - 1| \leq Cn^{-2/3}.$$

For the zeros x_{in}, $i = 1, 2, \ldots, n$, of $P_n(W_{rQ}^2; x)$ we have the following estimates.

Theorem 1.4. *Uniformly for $2 \leq j \leq n - 1$, $n = 3, 4, 5, \ldots$*

$$x_{j-1,n} - x_{j+1,n} \sim (a_n/n)[\max\{n^{-2/3}, 1 - (|x_{jn}|/a_n)\}]^{-1/2}.$$

Remark 1.5. In fact, we can show that for $2 \leqslant j \leqslant n$, $n = 2, 3, 4, \ldots$

$$x_{j-1,n} - x_{jn} \sim (a_n/n)[\max\{n^{-2/3}, 1 - (|x_{jn}|/a_n)\}]^{-1/2}.$$

The following expression of $P_n'(x)$ is worth our application.

Theorem 1.6. *We have an expression*

$$P_n'(x) = A_n(x)P_{n-1}(x) - B_n(x)P_n(x) - 2r\{P_n(x)/x\}^*,$$

where

$$A_n(x) = 2b_n \int_{-\infty}^{\infty} P_n^2(t)\bar{Q}(x,t)W_{rQ}^2(t)\,dt,$$

$$B_n(x) = 2b_n \int_{-\infty}^{\infty} P_n(t)P_{n-1}(t)\bar{Q}(x,t)W_{rQ}^2(t)\,dt,$$

$$\{P_n(x)/x\}^* = \begin{cases} P_n(x)/x & (n\text{: odd}), \\ 0 & (n\text{: even}), \end{cases}$$

$$\bar{Q}(x,t) = \{Q'(t) - Q'(x)\}/(t-x).$$

We estimate $A_n(x)$ and $B_n(x)$.

Theorem 1.7. *We have*

$$A_n(x) \sim n/a_n, \quad |B_n(x)| \leqslant C_n/a_n \quad \text{for } |x| \leqslant Da_n \ (D>0).$$

We define

$$(x)_r = \begin{cases} 0 & (r \geqslant 0), \\ x & (r<0). \end{cases}$$

The following is our main result:

Theorem 1.8. *For* $|x| \leqslant a_n(1 + Ln^{-2/3})$ *we have*

$$|P_n(x)W_Q(x)|\left(|x| + \left(\frac{a_n}{n}\right)_r\right)^r \leqslant Ca_n^{-1/2}\left[\max\left\{n^{-2/3}, 1 - \frac{|x|}{a_n}\right\}\right]^{-1/4}.$$

The values of $P_n'(x_{in})$, $i = 1, 2, \ldots, n$, are estimated as follows.

Theorem 1.9. (i) *If n is odd, then we have*

$$|P_{n-1}(0)| \sim (n/a_n)^r a_n^{-1/2}, \tag{1.6}$$

$$|P_n'(0)| \sim (n/a_n)^r n a_n^{-3/2}. \tag{1.7}$$

(ii) *For* $x_{jn} \neq 0$, *we have*

$$|(d/dx)\{P_n(x)W_{rQ}(x)\}_{x=x_{jn}}| = |P_n'(x_{jn})W_{rQ}(x_{jn})| \sim n a_n^{-3/2}\left[\max\left\{n^{-2/3}, 1-\frac{|x_{jn}|}{a_n}\right\}\right]^{1/4}, \tag{1.8}$$

especially we see

$$|(d/dx)\{P_n(x)W_{rQ}(x)\}_{x=x_{1n}}| = |P_n'(x_{1n})W_{rQ}(x_{1n})| \sim n a_n^{-3/2} n^{-1/6}.$$

We obtain an improvement of Theorem 1.4.

Theorem 1.10. *Uniformly for* $2 \leqslant j \leqslant n$, $n = 2, 3, 4, \ldots$, *we have*

$$C a_n/n \leqslant x_{j-1,n} - x_{jn},$$

especially for $|x_{jn}|$, $|x_{j-1,n}| \leqslant \eta a_n$, $0 < \eta < 1$, *we see*

$$x_{j-1,n} - x_{jn} \sim a_n/n.$$

Theorem 1.8 is improved as follows. Let $[x]$ denote the maximum integer nonexceeding x.

Theorem 1.11. *Let* $|x_{in}| \leqslant \eta a_n$, $0 < \eta < 1$.

(i) *We have*

$$\max_{|x| \leqslant x_{[n/2],n}} |P_n(x)| \sim (n/a_n)^r a_n^{-1/2}, \tag{1.9}$$

and if $0 < x_{kn}$ *or* $x_{k-1,n} < 0$, *then we have*

$$\max_{x_{kn} \leqslant x \leqslant x_{k-1,n}} |P_n(x)W_{rQ}(x)| \sim a_n^{-1/2}. \tag{1.10}$$

(ii) *We see*

$$\max_{|x| \leqslant x_{[n/2],n}} |P_n'(x)| \sim (n/a_n)^r n a_n^{-3/2}, \tag{1.11}$$

and if $0 < x_{kn}$ *or* $x_{k-1,n} < 0$, *then we have*

$$\max_{x_{kn} \leqslant x \leqslant x_{k-1,n}} |P_n'(x)W_{rQ}(x)| \sim n a_n^{-3/2}. \tag{1.12}$$

The following precision of Theorem 1.11 is applicable.

Corollary 1.12. *Let* $|x_{in}| \leqslant \eta a_n, \; 0 < \eta < 1$.

(i) *Let n be odd. For* $0 < \delta a_n/n \leqslant x \leqslant x_{[n/2],n} - \delta a_n/n, \; \delta > 0$, *we see*

$$|P_n(x)| \sim (n/a_n)^r a_n^{-1/2},$$

and there is a constant $\delta' > 0$ *such that for* $|x| \leqslant \delta' a_n/n$

$$|P_n'(x)| \sim (n/a_n)^r n a_n^{-3/2}.$$

Let n be even. For $-x_{[n/2],n} + \delta a_n/n \leqslant x \leqslant x_{[n/2],n} - \delta a_n/n, \; \delta > 0$, *we see*

$$|P_n(x)| \sim (n/a_n)^r a_n^{-1/2}.$$

(ii) *Otherwise for* $x_{kn} + \delta a_n/n \leqslant x \leqslant x_{k-1,n} - \delta a_n/n, \; \delta > 0$, *we see*

$$|P_n(x) W_{rQ}(x)| \sim a_n^{-1/2},$$

and there is a constant $\delta' > 0$ *such that for* $x_{kn} - \delta' a_n/n \leqslant x \leqslant x_{kn} + \delta' a_n/n$

$$|P_n'(x) W_{rQ}(x)| \sim n a_n^{-3/2}.$$

The followings are related to the maximum value for $P_n(x) W_{rQ}(x)$ on $\mathbf{R}$.

Theorem 1.13. *Let Q satisfy* (0.1). *We have*

$$\sup_{x \in \mathbf{R}} |P_n(x) W_Q(x)| \left(|x| + \left(\frac{a_n}{n} \right)_r \right)^r \left| 1 - \frac{|x|}{a_n} \right|^{1/4} \sim a_n^{-1/2}.$$

Theorem 1.14. *Let Q satisfy* (0.1), *and* $Q(0) = 0$. *We have*

$$\sup_{x \in \mathbf{R}} |P_n(x) W_Q(x)| \left(|x| + \left(\frac{a_n}{n} \right)_r \right)^r \sim a_n^{-1/2} n^{1/6}.$$

Theorem 1.15. *Let Q satisfy*(0.1), $Q(0) = 0$ *and let* $r \geqslant 0$. *We have*

$$\sup_{x \in \mathbf{R}} |P_n'(x) W_{rQ}(x)| \sim n a_n^{-3/2} n^{1/6}.$$

2. Lemmas and proofs of theorems

To show Theorem 1.1 we need the following lemmas. Let $0 < p \leqslant \infty$.

Lemma 2.1 (Bauldry [Ba1, Theorem 3.1]). *Let $pr+1>0$. There are $C, D>0$ and n_0 such that for $n>n_0$ and $P\in\prod_n$*

$$||PW_{rQ}||_{L_p(\mathbf{R})}\leqslant C||PW_{rQ}||_{L_p[-Da_n,Da_n]}.$$

Lemma 2.2 (Levin and Lubinsky [LL1, Theorem 1.8]). *Let $K>0$. Then there exist n_0 and $C>0$ such that for $n\geqslant n_0$ and $P\in\prod_n$*

$$||PW_Q||_{L_p(\mathbf{R})}\leqslant C||PW_Q||_{L_p(|x|\leqslant a_n(1-Kn^{-2/3}))}.$$

Proof of Theorem 1.1. Let $P\in\prod_n$, and let us take $K'(>K)$ large enough. Then from Lemma 2.1 there is a constant $D>0$ such that

$$\begin{aligned}||PW_{rQ}||^p_{L_p(\mathbf{R})}&\leqslant C||PW_{rQ}||^p_{L_p[-Da_n,Da_n]}\\&\leqslant C\{||PW_{rQ}||^p_{L_p(|x|\leqslant a_n(1-K'n^{-2/3}))}+||PW_{rQ}||^p_{L_p(a_n(1-K'n^{-2/3})\leqslant|x|\leqslant Da_n)}\}.\end{aligned}$$

The second term in the last line is able to be estimated as follows. From Lemma 2.2 we have

$$\begin{aligned}&||PW_{rQ}||^p_{L_p(a_n(1-K'n^{-2/3})\leqslant|x|\leqslant Da_n)}\\&\quad=||Px^rW_Q||^p_{L_p(a_n(1-K'n^{-2/3})\leqslant|x|\leqslant Da_n)}\\&\quad\leqslant Ca_n^{(-[r+1]+r)p}||Px^{[r+1]}W_Q||^p_{L_p(a_n(1-K'n^{-2/3})\leqslant|x|\leqslant Da_n)}\\&\quad\leqslant Ca_n^{(-[r+1]+r)p}||Px^{[r+1]}W_Q||^p_{L_p(|x|\leqslant a_n(1-K'(n+[r+1])^{-2/3}))}\\&\quad\leqslant C||(x/a_n)^{[r+1]-r}PW_{rQ}||^p_{L_p(|x|\leqslant a_n(1-Kn^{-2/3}))}\\&\quad\leqslant C||PW_{rQ}||^p_{L_p(|x|\leqslant a_n(1-Kn^{-2/3}))}.\end{aligned}$$

Therefore, the proof of Theorem 1.1 is complete. □

We estimate $\lambda_n(W^2_{rQ};x)=\lambda_{n2}(x)$. To show it we need three lemmas.

Lemma 2.3 (Bauldry [Ba1, Corollary 3.4]). *Let $0<p<\infty$, and $pr>-1$, and let $D>0$ be the constant in Lemma 2.1. For every ε, $0<\varepsilon<1$, we have*

$$W^{-p}_{rQ}(x)\lambda_{np}(x)\sim(a_n/n)\left\{1+\frac{a_n}{n|x|}\right\}^{pr}\quad(|x|\leqslant\varepsilon Da_n).$$

Remark 2.4. This is given by Bauldry [Ba1]. Though he assumed the continuity of Q'' in $(-\infty,\infty)$, we can omit the continuity of Q'' at $x=0$. In fact, we can show that for fixed a and b there is a value $a<\xi<b$ such that $Q'(b)-Q'(a)=Q''(\xi)(b-a)$ (see [Ba1, Lemma 5.2]).

Lemma 2.5. *We have*

$$\max\{(n+r)^{-2/3}, 1-(|x|/a_{n+r})\} \sim \max\{n^{-2/3}, 1-(|x|/a_n)\} \quad (x \in \mathbf{R}).$$

Proof. We assume $r \geqslant 0$. When $r<0$, we can show the lemma similarly. From [LL1, Lemma 5.2(c)], for some fixed $\lambda > 1$ we have

$$|(a_u/a_v) - 1| \sim |(u/v) - 1| \tag{2.1}$$

uniformly for $v \in (0, \infty)$ and $u \in [v/\lambda, \lambda v]$. Especially (2.1) implies

$$1 - (a_n/a_{n+r}) \sim 1 - (n/(n+r)) = r/(n+r) = 0(1/n). \tag{2.2}$$

By (2.2), for $|x| \leqslant 2a_n$

$$\begin{aligned} 1-(|x|/a_{n+r}) &= 1-(|x|/a_n) + (|x|/a_n)\{1-(a_n/a_{n+r})\} \\ &= 1-(|x|/a_n) + 0(1/n). \end{aligned} \tag{2.3}$$

Obviously, we see $1-(|x|/a_n) < 1-(|x|/a_{n+r})$. Therefore, by (2.3) the lemma is true. □

Lemma 2.6 (Levin and Lubinsky [LL1, Theorem 1.1]). (i) *Given fixed* $L>0$, *we set* $J_n = \{t; |t| \leqslant a_n(1+Ln^{-2/3})\}$.

$$\lambda_n(W_Q^2; x) \sim (a_n/n) W_Q^2(x)[\max\{n^{-2/3}, 1-(|x|/a_n)\}]^{-1/2}$$

uniformly for $x \in J_n$ *and* $n \geqslant 1$.

(ii) *For all* $x \in \mathbf{R}$ *and* $n \geqslant 1$

$$\lambda_n(W_Q^2; x) \geqslant C(a_n/n) W_Q^2(x)[\max\{n^{-2/3}, 1-(|x|/a_n)\}]^{-1/2}.$$

Proof of Theorem 1.2. (i) In Lemma 2.3, we only put $p=2$.

(ii) Let $\varepsilon a_n/n \leqslant |x| \leqslant \eta a_n$, $0<\eta<1$. If we put $p=2$ in Lemma 2.3, then we have

$$\lambda_n(W_{rQ}^2; x) \geqslant C(a_n/n) W_{rQ}^2(x).$$

We show the theorem for $\eta a_n \leqslant |x|$.

$$\lambda_n(W_{rQ}^2; x) = \inf_{P \in \prod_{n-1}} \int_{-\infty}^{\infty} (PW_{rQ})^2(t)\, dt / P^2(x)$$

$$
\begin{aligned}
&\geqslant \inf_{P\in\prod_{n-1}}\left[\int_{|t|\leqslant a_n}\frac{\{t^{[r+1]}|x/t|^{[r+1]-r}(PW_Q)(t)\}^2\,dt}{\{|x|^{[r+1]}P(x)\}^2}\right]|x|^{2r}\\
&\geqslant C\inf_{P\in\prod_{n+[r+1]-1}}\left[\int_{|t|\leqslant a_{n+[r+1]-1}}(PW_Q)^2(t)\,dt/P^2(x)\right]|x|^{2r}\\
&\geqslant C\{a_{n+[r+1]}/(n+[r+1])\}W_{rQ}^2(x)\\
&\quad\times[\max\{(n+[r+1])^{-2/3},1-(|x|/a_{n+[r+1]})\}]^{-1/2}\\
&\quad\text{(by Lemmas 2.2 and 2.6(ii))}\\
&\geqslant C(a_n/n)W_{rQ}^2(x)[\max\{n^{-2/3},1-(|x|/a_n)\}]^{-1/2}\quad\text{(by Lemma 2.5).}
\end{aligned}
$$

(iii) Let $\varepsilon a_n/n\leqslant|x|\leqslant\eta a_n,\ 0<\eta<1$. Then we put $p=2$ in Lemma 2.3. We show the theorem for $\eta a_n\leqslant|x|\leqslant a_n(1+Ln^{-2/3})$. Using Lemmas 2.1 and 2.6(i)

$$
\begin{aligned}
\lambda_n(W_{rQ}^2;x)&=\inf_{P\in\prod_{n-1}}\int_{-\infty}^{\infty}(PW_{rQ})^2(t)\,dt/P^2(x)\\
&\leqslant C\inf_{P\in\prod_{n-1}}\left\{\int_{|t|\leqslant Da_n}\{|t|^r(PW_Q)(t)\}^2\,dt/P^2(x)\right\}\\
&\leqslant Ca_n^{2r}\inf_{P\in\prod_{n-1}}\left\{\int_{-\infty}^{\infty}(PW_Q)^2(t)\,dt/P^2(x)\right\}\\
&\leqslant Ca_n^{2r}(a_n/n)W_Q^2(x)[\max\{n^{-2/3},1-(|x|/a_n)\}]^{-1/2}\\
&\leqslant C(a_n/n)W_{rQ}^2(x)[\max\{n^{-2/3},1-(|x|/a_n)\}]^{-1/2}\quad(\text{by } a_n\sim|x|).
\end{aligned}
$$

The inverse inequality follows from (ii). From these results, the proof is complete. □

Lemma 2.7. *We assume that $pr+1>0$ if $0<p<\infty$, and $r\geqslant 0$ if $p=\infty$. There exist constants $\varepsilon, C>0$ such that for every $P\in\prod_n$ and $n=0,1,2,\ldots$, we have*

$$||PW_{rQ}||_{L_p(|x|\leqslant\varepsilon a_n/n)}\leqslant C||PW_{rQ}||_{L_p(\varepsilon a_n/n\leqslant|x|\leqslant a_n)},$$

where $0<p\leqslant\infty$. Especially, for $r=0$, we have

$$||P||_{L_p(|x|\leqslant\varepsilon a_n/n)}\leqslant C||P||_{L_p(\varepsilon a_n/n\leqslant|x|\leqslant a_n)}.$$

Proof. We use the estimates on the L_p Christoffel functions. By the definition we have for all x and all P of degree $\leqslant n$,

$$|PW_{rQ}|^p(x)\leqslant\lambda_{np}^{-1}(x)W_{rQ}^p(x)\int_{-\infty}^{\infty}|PW_{rQ}|^p(x)\,dx.$$

Using our estimates for Christoffel functions from Lemma 2.3, and the inequality Theorem 1.1, we obtain, for $\varepsilon > 0$, and some C independent of n, P, ε,

$$\begin{aligned}\int_{-\varepsilon a_n/n}^{\varepsilon a_n/n} |PW_{rQ}|^p(x)\,dx &\leqslant C\left[\int_{-\varepsilon a_n/n}^{\varepsilon a_n/n} \frac{n}{a_n}\left(1+\frac{a_n}{n|x|}\right)^{-pr} dx\right]\int_{-a_n}^{a_n} |PW_{rQ}|^p(x)\,dx \\ &= 2C\left[\int_0^{\varepsilon}\left(1+\frac{1}{t}\right)^{-pr} dt\right]\int_{-a_n}^{a_n} |PW_{rQ}|^p(x)\,dx \\ &\leqslant C\varepsilon^{pr+1}\int_{-a_n}^{a_n} |PW_{rQ}|^p(x)\,dx,\end{aligned}$$

where C is independent of n, P, ε. Then we deduce that

$$\left(\int_{-\varepsilon a_n/n}^{\varepsilon a_n/n} |PW_{rQ}|^p(x)\,dx\right)(1-C\varepsilon^{pr+1}) \leqslant C\varepsilon^{pr+1}\int_{\varepsilon\frac{a_n}{n}\leqslant|x|\leqslant a_n} |PW_{rQ}|^p(x)\,dx,$$

that is,

$$\begin{aligned}&\left(\int_{-\varepsilon a_n/n}^{\varepsilon a_n/n} |PW_{rQ}|^p(x)\,dx\right)^{1/p}(1-C\varepsilon^{pr+1})^{1/p} \\ &\quad\leqslant C\varepsilon^{r+1/p}\left(\int_{\varepsilon\frac{a_n}{n}\leqslant|x|\leqslant a_n} |PW_{rQ}|^p(x)\,dx\right)^{1/p}.\end{aligned}$$

So, for $0 < p < \infty$ the lemma follows if we choose ε small enough. Furthermore, when $r \geqslant 0$ and ε is small enough, let $p \to \infty$, then we have the lemma for $p = \infty$. □

Proof of Theorem 1.3. We follow the method of [LL1, Proof of Corollary 1.2(a)]. For more general weight $W(x)$ we have (see e.g. [Fr2])

$$x_{1n} = \sup_{p\in\prod_{2n-2}, P\geqslant 0} \frac{\int_{-\infty}^{\infty} xP(x)W^2(x)\,dx}{\int_{-\infty}^{\infty} P(x)W^2(x)\,dx}.$$

Especially, we apply it for the weight $W_{rQ}(x)$,

$$a_n - x_{1n} = \inf_{p\in\prod_{2n-2}, P\geqslant 0} \frac{\int_{-\infty}^{\infty} (a_n-x)P(x)W_{rQ}^2(x)\,dx}{\int_{-\infty}^{\infty} P(x)W_{rQ}^2(x)\,dx}.$$

We see that nth Mhaskar–Rahmanov–Saff number $\bar{a}_n$ for W_Q^2 satisfies $\bar{a}_n = a_{n/2}$. Therefore, when we use Theorem 1.1 with respect to W_{rQ} in L_1-space, for a_n we may take an integrant polynomial of degree $2n$. From this

$$|a_n - x_{1n}| \sim \inf_{P\in\prod_{2n-2}, P\geqslant 0} \frac{\int_{|x|\leqslant a_n(1-Ln^{-2/3})} (a_n-x)P(x)W_{rQ}^2(x)\,dx}{\int_{|x|\leqslant a_n} P(x)W_{rQ}^2(x)\,dx}.$$

We set $m = [n^{1/3}]$ (Gaussian symbol), and $P(x) = \lambda_{n-m}^{-1}(x)R^2(x)$, $R\in\prod_m$. We see that $|x| \leqslant a_n$ means $|x| \leqslant a_{n-m}(1+L(n-m)^{-2/3})$ for n large enough, because $a_n/a_{n-m} - 1 \sim n/(n-m) - 1 = O(n^{-2/3})$, by (2.1). From Theorem 1.2(iii) we obtain that for $\varepsilon a_n/n \leqslant |x| \leqslant a_n$

$$\lambda_{n-m}^{-1}(W_{rQ}^2 : x)W_{rQ}^2(x) \sim \{(n-m)/a_{n-m}\}[\max\{(n-m)^{-2/3}, 1-(|x|/a_{n-m})\}]^{1/2}.$$

From (2.1) we see $a_n/a_{n-m} = 1 + 0(n^{-2/3})$, hence

$$\begin{aligned}1-(|x|/a_n) &= 1-(|x|/a_{n-m}) + (|x|/a_{n-m})\{1-(a_{n-m}/a_n)\}\\ &= 1-(|x|/a_{n-m}) + 0(n^{-2/3}).\end{aligned}$$

Therefore,

$$\lambda_{n-m}^{-1}(W_{rQ}^2 : x)W_{rQ}^2(x) \sim (n/a_n)[\max\{n^{-2/3}, 1-(|x|/a_n)\}]^{1/2}.$$

For $|x| \leqslant \varepsilon a_n/n$ we have

$$\lambda_{n-m}^{-1}(x)W_{rQ}^2(x) \leqslant C(n/a_n)^{2r+1}|x|^{2r} \leqslant \varepsilon^{2r}C(n/a_n)\{1-(|x|/a_n)\}^{1/2}$$

(by Theorem 1.2(i)). From these results we see

$$\begin{aligned}&\int_{|x|\leqslant a_n(1-Ln^{-2/3})} (a_n-x)\lambda_{n-m}^{-1}(x)R^2(x)W_{rQ}^2(x)\,dx\\ &\quad\leqslant \int_{|x|\leqslant a_n(1-Ln^{-2/3})} (a_n-x)(n/a_n)\{1-(|x|/a_n)\}^{1/2}R^2(x)\,dx.\end{aligned}$$

On the other hand, for $\varepsilon a_n/n \leqslant |x| \leqslant a_n$ we see

$$\lambda_{n-m}^{-1}(x)W_{rQ}^2(x) \geqslant C(n/a_n)\{1-(|x|/a_n)\}^{1/2}.$$

Therefore,

$$\begin{aligned}&\int_{|x|\leqslant a_n} \lambda_{n-m}^{-1}(x)R^2(x)W_{rQ}^2(x)\,dx\\ &\quad\geqslant C\int_{\varepsilon a_n/n\leqslant|x|\leqslant a_n} (n/a_n)\{1-(|x|/a_n)\}^{1/2}R^2(x)\,dx.\end{aligned}$$

Let $|x| \leqslant \varepsilon a_n/n$, where $\varepsilon > 0$ is small enough and independent of n. By Lemma 2.7 we have for some constant C_1

$$\begin{aligned}&\int_{|x|\leqslant\varepsilon a_n/n} (n/a_n)\{1-(|x|/a_n)\}^{1/2}R^2(x)\,dx\\ &\quad\leqslant C_1\int_{\varepsilon a_n/n\leqslant|x|\leqslant a_n} (n/a_n)\{1-(|x|/a_n)\}^{1/2}R^2(x)\,dx.\end{aligned}$$

Hence,

$$\begin{aligned}&\int_{|x|\leqslant a_n} \lambda_{n-m}^{-1}(x)R^2(x)W_{rQ}^2(x)\,dx\\ &\quad\geqslant \frac{C}{C_1+1}\int_{|x|\leqslant a_n} (n/a_n)\{1-(|x|/a_n)\}^{1/2}R^2(x)\,dx.\end{aligned}$$

Consequently,

$$\begin{aligned}
|a_n - x_{1n}| &\leqslant C \inf_{R\in \prod_m} \frac{\int_{|x|\leqslant a_n(1-Ln^{-2/3})} (a_n - x)\{1-(\frac{|x|}{a_n})\}^{1/2} R^2(x)\,dx}{\int_{|x|\leqslant a_n}\{1-(\frac{|x|}{a_n})\}^{1/2} R^2(x)\,dx}\\
&\leqslant Ca_n \inf_{S\in \prod_m} \frac{\int_{-1}^{1} \{(1-s)S^2(s)(1-|s|)^{1/2}\}\,ds}{\int_{-1}^{1} \{S^2(s)(1-|s|)^{1/2}\}\,ds}\\
&\leqslant Ca_n \inf_{S\in \prod_m} \frac{\int_{-1}^{1} \{(1-s)S^2(s)(1-s^2)^{1/2}\}\,ds}{\int_{-1}^{1} \{S^2(s)(1-s^2)^{1/2}\}\,ds}\\
&= Ca_n(1-x^*_{1,m+1}).
\end{aligned}$$

Here $x^*_{1,m+1}$ is the largest zero of the $(m+1)$th orthonormal polynomial for the ultraspherical weight $(1-s^2)^{1/2}$ on $[-1,1]$. Since $1-x^*_{1,m+1}\leqslant Cm^{-2}$ by Szegö [Sz, Theorem 6.21.2], we see

$$|a_n - x_{1n}|\leqslant Ca_n m^{-2}\leqslant Ca_n n^{-2/3},$$

consequently we have $|(x_{1n}/a_n)-1|\leqslant Cn^{-2/3}$. □

To prove Theorem 1.4 we need the following lemmas.

Lemma 2.8. *Let us consider the zero $x_{[(n-1)/2],n}$. Then there is a constant $C>0$ such that $Ca_n/n\leqslant x_{[(n-1)/2],n}$. We note that $x_{[(n-1)/2],n}$ is the smallest positive zero if n is odd.*

Proof. By the Markov–Stieltjes inequality [Fr1, p. 33 (5.10)].

$$\begin{aligned}
\int_{-\infty}^{x_{i+1,n}} W_{rQ}^2(t)\,dt &\leqslant \sum_{k=i+1}^{n} \lambda_{kn} \leqslant \int_{-\infty}^{x_{in}} W_{rQ}^2(t)\,dt\\
&\leqslant \sum_{k=i}^{n} \lambda_{kn} \leqslant \int_{-\infty}^{x_{i-1,n}} W_{rQ}^2(t)\,dt.
\end{aligned}$$

Therefore,

$$\lambda_{in}\leqslant \int_{x_{i+1,n}\leqslant t\leqslant x_{i-1,n}} W_{rQ}^2(t)\,dt, \quad i=2,3,\ldots,n-1.$$

Let n be odd, and let us consider $x_{[(n+1)/2],n}=0$. Then using Theorem 1.2(i)

$$C(a_n/n)^{2r+1}\leqslant \lambda_{[(n+1)/2],n}\leqslant 2C(x_{[(n-1)/2],n})^{2r+1}. \tag{2.4}$$

From this we see $C(a_n/n)\leqslant 2x_{[(n-1)/2],n}$. Consequently, if n is odd, we have the lemma.

If n is even, then $0<x_{[(n-1)/2],n-1}<x_{[(n-1)/2],n}$. Hence, the lemma is complete. □

Lemma 2.9. *Let $x_{j+1,n}\geqslant x_{[(n-1)/2],n}$, then $1<x_{jn}/x_{j+1,n}\leqslant C$ for a constant C.*

Proof. By Lemma 2.8 we see $Ca_n/n \leqslant x_{j+1,n}$. Let $K>0$ be large enough. If $x_{jn} \leqslant Ka_n/n$, then we see $1 < x_{jn}/x_{j+1,n} \leqslant K/C$. Let $0<\eta<1$. If $\eta a_n \leqslant x_{j+1,n} < x_{jn} \leqslant a_n(1+Ln^{-2/3})$, then we see $1 < x_{jn}/x_{j+1,n} \leqslant 2/\eta \leqslant C$ for n large enough. Therefore, for $K>0$ large enough, and $0<\eta<1$, we may suppose

$$Ka_n/n \leqslant x_{j+1,n} < x_{jn} \leqslant \eta a_n. \tag{2.5}$$

Then for $x \in [Ka_n/n, \eta a_n]$ we see that, by Theorem 1.2(iii),

$$\lambda_n(W_{rQ}^2 : x) W_{rQ}^{-2}(x) \sim a_n/n. \tag{2.6}$$

Here by Lubinsky [Lu1,Lu2] we have an even positive entire function G with nonnegative Maclaurin series coefficients such that

$$G(x) \sim W_Q^{-2}(x), \quad x \in \mathbf{R}. \tag{2.7}$$

Then by the Posse–Markov–Stieltjes inequality [KL, Lemma 3.2], for $x_{j+1,n} > 0$

$$\begin{aligned}
&\lambda_{jn} G(x_{jn}) + \lambda_{j+1,n} G(x_{j+1,n}) \\
&\quad = (1/2)\left\{ \sum_{k:|x_{kn}|<x_{j-1,n}} \lambda_{kn} G(x_{kn}) - \sum_{k:|x_{kn}|<x_{j+1,n}} \lambda_{kn} G(x_{kn}) \right\} \\
&\quad \geqslant (1/2)\left\{ \int_{|t|\leqslant x_{jn}} - \int_{|t|\leqslant x_{j+1,n}} \right\} G(t) W_{rQ}^2(t)\, dt \\
&\quad = (1/2) \int_{x_{j+1,n}\leqslant t \leqslant x_{jn}} G(t) W_{rQ}^2(t)\, dt \geqslant C(x_{jn}^{2r+1} - x_{j+1,n}^{2r+1}) \quad (\text{by } (2.7)).
\end{aligned} \tag{2.8}$$

Formula (2.8) implies

$$x_{jn}^{2r} \lambda_{jn} W_{rQ}^{-2}(x_{jn}) + x_{j+1,n}^{2r} \lambda_{j+1,n} W_{rQ}^{-2}(x_{j+1,n}) \geqslant C(x_{jn}^{2r+1} - x_{j+1,n}^{2r+1}) \quad (\text{by } (2.7)).$$

From (2.6)

$$(a_n/n)\{(1/x_{jn})(x_{jn}/x_{j+1,n})^{2r+1} + (1/x_{j+1,n})\} \geqslant C\{(x_{jn}/x_{j+1,n})^{2r+1} - 1\}.$$

So

$$C + (a_n/n)(1/x_{j+1,n}) \geqslant \{C - (a_n/n)(1/x_{jn})\}(x_{jn}/x_{j+1,n})^{2r+1}.$$

By (2.5) we see $Ka_n/n \leqslant x_{j+1,n} < x_{jn}$, hence we have

$$C + (1/K) \geqslant \{C - (1/K)\}(x_{jn}/x_{j+1,n})^{2r+1}.$$

Consequently, we have $1 < x_{jn}/x_{j+1,n} \leqslant C$. $\square$

Proof of Theorem 1.4. We take an even positive entire function G as (2.7), that is, $G(x) \sim W_Q^{-2}(x)$, $x \in \mathbf{R}$. By the new Posse–Markov–Stieltjes inequality used in (2.8), we see that for $2 \leqslant j \leqslant n-1$

$$\begin{aligned}\lambda_{jn} G(x_{jn}) &= (1/2)\left[\sum_{k\,:\,|x_{kn}|<|x_{j-1,n}|} \lambda_{kn} G(x_{kn}) - \sum_{k\,:\,|x_{kn}|<|x_{jn}|} \lambda_{kn} G(x_{kn})\right] \\ &\leqslant (1/2)\left\{\int_{|t|\leqslant|x_{j-1,n}|} - \int_{|t|\leqslant|x_{j+1,n}|}\right\} G(t) W_{rQ}^2(t)\,dt \\ &= (1/2)\int_{|x_{j+1,n}|\leqslant t\leqslant|x_{j-1,n}|} G(t) W_{rQ}^2(t)\,dt.\end{aligned}$$

If $x_{jn} = 0$, then by Theorem 1.2(i)

$$(a_n/n)^{2r+1} \sim \lambda_{jn} G(x_{jn}) \leqslant C(x_{j-1,n}^{2r+1} - x_{j+1,n}^{2r+1}) = 2C(x_{[(n-1)/2],n})^{2r+1}.$$

From this we have

$$Ca_n/n \leqslant 2x_{[(n-1)/2],n} = x_{j-1,n} - x_{j+1,n}.$$

Since $x_{jn} = 0$, we see

$$C(a_n/n)\max\{n^{-2/3}, 1-(|x_{jn}|/a_n)\}^{-1/2} \leqslant x_{j-1,n} - x_{j+1,n}.$$

Let $x_{j+1,n} = 0$. By Lemma 2.9 we see $x_{jn} \sim x_{j-1,n}$, so we have

$$\begin{aligned}\lambda_{jn} W_{rQ}^{-2}(x_{jn}) \sim \lambda_{jn} G(x_{jn})/x_{jn}^{2r} &\leqslant C(1/x_{jn}^{2r})(x_{j-1,n}^{2r+1} - x_{j+1,n}^{2r+1}) \\ &\leqslant C(1/x_{jn}^{2r}) x_{j-1,n}^{2r}(x_{j-1,n} - x_{j+1,n}) \\ &\leqslant C(x_{j-1,n} - x_{j+1,n}).\end{aligned}$$

From Theorem 1.2(ii)

$$C(a_n/n)[\max\{n^{-2/3}, 1-(|x_{jn}|/a_n)\}]^{-1/2} \leqslant (x_{j-1,n} - x_{j+1,n}).$$

For $x_{j-1,n} = 0$ we have the same result.

Let $x_{j+1,n}, x_{jn}, x_{j-1,n} > 0$. By Theorem 1.2(ii) and Lemma 2.9

$$\begin{aligned}&C(a_n/n)[\max\{n^{-2/3}, 1-(|x_{jn}|/a_n)\}]^{-1/2} \\ &\quad\leqslant \lambda_{jn} W_{rQ}^{-2}(x_{jn}) \sim \lambda_{jn} G(x_{jn})/x_{jn}^{2r} \\ &\quad\leqslant C(1/x_{jn}^{2r})(x_{j-1,n}^{2r+1} - x_{j+1,n}^{2r+1}) \\ &\quad\leqslant C(1/x_{jn}^{2r}) x_{j-1,n}^{2r}(x_{j-1,n} - x_{j+1,n}) \leqslant C(x_{j-1,n} - x_{j+1,n}).\end{aligned}$$

We show the inverse inequality. From (2.8)

$$\lambda_{jn} G(x_{jn}) + \lambda_{j+1,n} G(x_{j+1,n}) \geqslant \int_{x_{j+1,n}\leqslant t\leqslant x_{jn}} G(t) W_{rQ}^2(t)\,dt.$$

Let $x_{j+1,n} = 0$. From Theorem 1.2(i) and (iii)

$$x_{jn}^{2r}(a_n/n)[\max\{n^{-2/3}, 1-(|x_{jn}|/a_n)\}]^{-1/2} + (a_n/n)^{2r+1} \geqslant Cx_{jn}^{2r+1}.$$

Therefore,

$$(a_n/n)[\max\{n^{-2/3}, 1-(|x_{jn}|/a_n)\}]^{-1/2} + (1/x_{jn}^{2r})(a_n/n)^{2r+1} \geqslant Cx_{jn}.$$

Since by Lemma 2.8 we see $Ca_n/n \leqslant x_{jn}$, we have

$$(a_n/n)[\max\{n^{-2/3}, 1-(|x_{jn}|/a_n)\}]^{-1/2} \geqslant Cx_{jn} \geqslant Cx_{j-1,n} \geqslant C(x_{j-1,n} - x_{j+1,n}).$$

Similarly, for $x_{jn} = 0$ we can show the same inequality. Let $x_{jn}, x_{j+1,n} > 0$. Then

$$\begin{aligned} x_{jn}^{2r}\lambda_{jn}W_{rQ}^{-2}(x_{jn}) + x_{j+1,n}^{2r}\lambda_{j+1,n}W_{rQ}^{-2}(x_{j+1,n}) &\geqslant C(x_{jn}^{2r+1} - x_{j+1,n}^{2r+1}) \\ &\geqslant Cx_{jn}^{2r}(x_{jn} - x_{j+1,n}). \end{aligned}$$

By Theorem 1.2(iii)

$$\begin{aligned} &C(a_n/n)\{[\max\{n^{-2/3}, 1-(|x_{jn}|/a_n)\}]^{-1/2} \\ &\quad + [\max\{n^{-2/3}, 1-(x_{j+1,n}/a_n)\}]^{-1/2}\} \geqslant (x_{jn} - x_{j+1,n}). \end{aligned}$$

From this we have

$$C(a_n/n)[\max\{n^{-2/3}, 1-(x_{jn}/a_n)\}]^{-1/2} \geqslant x_{jn} - x_{j+1,n}. \tag{2.9}$$

This inequality also means

$$C(a_n/n)[\max\{n^{-2/3}, 1-(x_{j-1,n}/a_n)\}]^{-1/2} \geqslant x_{j-1,n} - x_{jn}. \tag{2.10}$$

Here we see

$$[\max\{n^{-2/3}, 1-(x_{jn}/a_n)\}]^{-1/2} \sim [\max\{n^{-2/3}, 1-(x_{j-1,n}/a_n)\}]^{-1/2}. \tag{2.11}$$

In fact, if $1-(x_{j-1,n}/a_n) \leqslant n^{-2/3}$, then

$$\begin{aligned} 1 - n^{-2/3} &\leqslant x_{j-1,n}/a_n = x_{jn}/a_n + (x_{j-1,n} - x_{jn})/a_n \\ &\leqslant x_{jn}/a_n + Cn^{-2/3} \quad (\text{by } (2.10)), \end{aligned}$$

that is, $1-(x_{jn}/a_n) < Cn^{-2/3}$. If $1-(x_{j-1,n}/a_n) > n^{-2/3}$, then by (2.10)

$$1 < \frac{1-(x_{jn}/a_n)}{1-(x_{j-1,n}/a_n)} = 1 + \frac{(x_{j-1,n} - x_{jn})/a_n}{1-(x_{j-1,n}/a_n)} \leqslant 1 + C.$$

Therefore we have (2.11). Consequently, from (2.9) and (2.10) we obtain

$$C(a_n/n)[\max\{n^{-2/3}, 1-(x_{jn}/a_n)\}]^{-1/2} \geqslant x_{j-1,n} - x_{j+1,n}.$$

The proof of the theorem is complete. □

Proof of Theorem 1.6. Using the reproducing kernel (1.1)

$$\begin{aligned}
P_n'(x) &= \int_{-\infty}^{\infty} P_n'(t)K_n(x,t)W_{rQ}^2(t)\,dt \\
&= -\int_{-\infty}^{\infty} P_n(t)K_n(x,t)\{2rt^{-1} - 2Q'(t)\}W_{rQ}^2(t)\,dt \\
&= 2\int_{-\infty}^{\infty} P_n(t)K_n(x,t)Q'(t)W_{rQ}^2(t)\,dt \\
&\quad - 2r\int_{-\infty}^{\infty} P_n(t)K_n(x,t)(1/t)W_{rQ}^2(t)\,dt \\
&= \int_1 - 2r\int_2 .
\end{aligned}$$

Here

$$\begin{aligned}
\int_1 &= -2b_n\int_{-\infty}^{\infty} P_n(t)\{P_n(x)P_{n-1}(t) - P_n(t)P_{n-1}(x)\}\bar{Q}(x,t)W_{rQ}^2(t)\,dt \\
&= 2b_n\left\{\int_{-\infty}^{\infty} P_n^2(t)\bar{Q}(x,t)W_{rQ}^2(t)\,dt\right\}P_{n-1}(x) \\
&\quad - 2b_n\left\{\int_{-\infty}^{\infty} P_n(t)P_{n-1}(t)\bar{Q}(x,t)W_{rQ}^2(t)\,dt\right\}P_n(x) \\
&= A_n(x)P_{n-1}(x) - B_n(x)P_n(x), \quad \int_2 = \int_{-\infty}^{\infty}\{P_n(t)/t\}K_n(x,t)W_{rQ}^2(t)\,dt.
\end{aligned}$$

If n is odd, then $P_n(t)/t \in \prod_{n-1}$. So we have

$$2r\int_2 = 2rP_n(x)/x.$$

Let n be even. Then $\int_2$ must be interpreted as a Cauchy principal value integral. Moreover, as the integral of an odd function over a symmetric interval is 0: $\int_{-\infty}^{\infty}$ (odd function) $dx = 0$. Therefore, we see

$$\int_2 = \int_{-\infty}^{\infty} P_n(t)\sum_{k\,:\,\mathrm{odd},k<n} P_k(x)\{P_k(t)/t\}W_{rQ}^2(t)\,dt = 0$$

(by the orthogonality). □

To prove Theorem 1.7 we need some lemmas. In the course of proving Theorem 1.7, we will show Theorem 1.8.

Lemma 2.10 (Levin and Lubinsky [LL1, Lemma 12.1]). *Let* $\rho \in (0, \min\{1, A-1\})$ *and* $\alpha \in (1, \{1-\rho\}^{-1})$. *Then for* $|x| \leqslant Da_n$ $(D \geqslant 1)$ *and* $n \geqslant 1$, *we have*

$$\int_{|t| \leqslant a_n} \bar{Q}(x,t)^{\alpha}\, dt \leqslant C a_n (n/a_n^2)^{\alpha}.$$

Lemma 2.11 (cf. Levin and Lubinsky [LL1, Lemma 12.2]). *Let us define for* $n \geqslant 1$

$$\chi_n = \max\left\{1, \max_{|t| \leqslant a_n/2} a_n P_n^2(t) W_Q^2(t)(|t| + (a_n/n)_r)^{2r}\right\},$$

and let ρ *and* α *be defined in Lemma* 2.10. *For* $|x| \leqslant Da_n$ $(D \geqslant 1)$ *we have*

$$A_n(x)/b_n \leqslant C(n/a_n^2)\chi_n^{1/\alpha}.$$

Proof. First, let $r \geqslant 0$. We repeat the method of [LL1]. Let $K \geqslant 2D \geqslant 4$. For $|x| \leqslant Da_n$ and $|t| \geqslant Ka_n$ we see $\bar{Q}(x,t) \sim Q'(t)/t \leqslant Q'(1)|t|^{B^*-2}$, where $B^* \geqslant 2$ is even integer [LL1, Lemma 5.1 (5.2)]. From this and [Ba1, p. 222]

$$\begin{aligned}
\int_{|t| \geqslant Ka_n} \{P_n(t) W_{rQ}(t)\}^2 \bar{Q}(x,t)\, dt &\leqslant C \int_{|t| \geqslant Ka_n} \{P_n(t)|t|^{\frac{B^*-2}{2}} W_{rQ}(t)\}^2\, dt \\
&\leqslant \exp(-Cn) \int_{|t| \leqslant Ka_n} \{P_n(t)|t|^{\frac{B^*-2}{2}} W_{rQ}(t)\}^2\, dt \\
&\leqslant \exp(-Cn)(Ka_n)^{(B^*-2)} \int_{-\infty}^{\infty} P_n^2(t) W_{rQ}^2(t)\, dt \\
&= o(a_n^{-2}).
\end{aligned}$$

From this we see

$$A_n(x)/b_n = \int_{|t| \leqslant Ka_n} \{P_n(t) W_{rQ}(t)\}^2 \bar{Q}(x,t)\, dt + o(a_n^{-2}).$$

Let $\theta = 2/\alpha$, $p = 2/(2-\theta)$, $q = 2/\theta = \alpha$, and $p^{-1} + q^{-1} = 1$. Then from $\alpha > 1$ we see $\theta < 2$. By the definition of χ_n we obtain

$$\begin{aligned}
&\int_{|t| \leqslant a_n/2} \{P_n(t) W_{rQ}(t)\}^2 \bar{Q}(x,t)\, dt \\
&\quad \leqslant (\chi_n/a_n)^{\theta/2} \int_{|t| \leqslant a_n/2} |P_n(t) W_{rQ}(t)\}^{2-\theta} \bar{Q}(x,t)\, dt \\
&\quad \leqslant (\chi_n/a_n)^{\theta/2} \left\{\int_{|t| \leqslant a_n/2} |P_n(t) W_{rQ}(t)|^{(2-\theta)p}\, dt\right\}^{1/p} \left\{\int_{|t| \leqslant a_n/2} \bar{Q}(x,t)^q\, dt\right\}^{1/q} \\
&\quad \leqslant (\chi_n/a_n)^{1/\alpha} \left\{\int_{|t| \leqslant a_n/2} \bar{Q}(x,t)^{\alpha}\, dt\right\}^{1/\alpha} \leqslant C\chi_n^{1/\alpha}(n/a_n^2) \quad \text{(by Lemma 2.10)}.
\end{aligned}$$

For $|x|\leqslant Da_n$ and $a_n/2\leqslant|t|\leqslant Ka_n$ we see $\bar{Q}(x,t)\leqslant CQ''(Ka_n)\leqslant Cn/a_n^2$ (see (0.1), (1.4)). Therefore,

$$\int_{a_n/2\leqslant|t|\leqslant Ka_n}\{P_n(t)W_{rQ}(t)\}^2\bar{Q}(x,t)\,dt\leqslant C(n/a_n^2)\chi_n^{1/\alpha}$$
(by the definition of χ_n).

Next, we assume $r<0$. Let θ, p and q be defined as above. Then, we see

$$\begin{aligned}
&\int_{|t|\leqslant a_n/2}\{P_n(t)W_{rQ}(t)\}^2\bar{Q}(x,t)\,dt\\
&\leqslant(\chi_n/a_n)^{\theta/2}\int_{|t|\leqslant a_n/2}|P_n(t)W_{rQ}(t)|^{2-\theta}\bar{Q}(x,t)\left(\frac{|t|}{|t|+a_n/n}\right)^{\theta r}dt\\
&\leqslant(\chi_n/a_n)^{\theta/2}\left\{\int_{|t|\leqslant a_n/2}|P_n(t)W_{rQ}(t)|^{(2-\theta)p}\,dt\right\}^{1/p}\\
&\quad\times\left\{\int_{|t|\leqslant a_n/2}\bar{Q}(x,t)^q\left(\frac{|t|}{|t|+a_n/n}\right)^{\theta rq}dt\right\}^{1/q}\\
&\leqslant(\chi_n/a_n)^{1/\alpha}\left\{\int_{|t|\leqslant a_n/2}\bar{Q}(x,t)^\alpha\left(\frac{|t|}{|t|+a_n/n}\right)^{2r}dt\right\}^{1/\alpha}.
\end{aligned}$$

Here, we will estimate

$$\int_{|t|\leqslant a_n/n}\bar{Q}(x,t)^\alpha\left(\frac{|t|}{|t|+a_n/n}\right)^{2r}dt\leqslant C\left(\frac{n}{a_n}\right)^{2r}\int_{|t|\leqslant a_n/n}\bar{Q}(x,t)^\alpha|t|^{2r}\,dt.$$

Let $|t|\leqslant a_n/n$ and $|x|\leqslant 2a_n/n$, then we see $\bar{Q}(x,t)=Q''(s)\leqslant C$ for some $s(|s|<2a_n/n)$, where C is a constant independent of n. If $2a_n/n\leqslant|x|$, then we have, for a certain $|s|\geqslant a_n/n$ and ρ in Lemma 2.10

$$\begin{aligned}
\bar{Q}(x,t)&=Q''(s)\leqslant CQ'(s)/s=C\frac{Q'(|s|)}{|s|^\rho}\times|s|^{\rho-1}\\
&\leqslant\frac{Q'(a_n)}{a_n}\left(\frac{n}{a_n}\right)^{1-\rho}\leqslant C\frac{n^{2-\rho}}{a_n^{3-\rho}}
\end{aligned}$$

by the monotonicity of $Q'(|s|)/|s|^\rho$ (see [LL1, proof of Lemma 12.1]). On the other hand, we see

$$\left(\frac{n}{a_n}\right)^{2r}\int_{|t|\leqslant a_n/n}|t|^{2r}\,dt\leqslant Ca_n/n.$$

Therefore,

$$\int_{|t|\leqslant a_n/n} \bar{Q}(x,t)^{\alpha}\left(\frac{|t|}{|t|+a_n/n}\right)^{2r} dt$$
$$\leqslant C\left(\frac{a_n}{n}\right)\left(\frac{n^{2-\rho}}{a_n^{3-\rho}}\right)^{\alpha} \leqslant C\left(\frac{n}{a_n^2}\right)^{\alpha}\left(\frac{n}{a_n}\right)^{(1-\rho)\alpha-1} \leqslant C\left(\frac{n}{a_n^2}\right)^{\alpha}$$

by $(1-\rho)\alpha-1<0$. So, we have

$$\int_{|t|\leqslant a_n/2} \{P_n(t)W_{rQ}(t)\}^2 \bar{Q}(x,t)\,dt \leqslant C(n/a_n^2)\chi_n^{1/\alpha}.$$

For the other part $\int_{|t|\geqslant a_n/2}$ we can repeat the same line as the case of $r\geqslant 0$. Consequently, we have the lemma. □

Proof of Theorem 1.8. By the Christoffel–Darboux formula (1.1) we see

$$P_n(x) = \{K_n(x,x_{kn})(x-x_{kn})\}/\{b_n P_{n-1}(x_{kn})\}.$$

Furthermore, by (1.2) and Theorem 1.6 we have

$$\lambda_n^{-1}(W_{rQ}^2;x_{kn}) = b_n P_n'(x_{kn})P_{n-1}(x_{kn}), \quad P_n'(x_{kn}) = A_n(x_{kn})P_{n-1}(x_{kn}). \tag{2.12}$$

Therefore, we have

$$\lambda_n^{-1}(W_{rQ}^2;x_{kn}) = b_n A_n(x_{kn})P_{n-1}^2(x_{kn}). \tag{2.13}$$

Applying the Cauchy–Schwarz inequality, we have

$$|P_n(x)| \leqslant \lambda_n^{-1/2}(x)\lambda_n^{-1/2}(x_{kn})|x-x_{kn}|/|b_n P_{n-1}(x_{kn})|$$
$$= \lambda_n^{-1/2}(x)\{A_n(x_{kn})/b_n\}^{1/2}|x-x_{kn}|.$$

Therefore, by Theorem 1.2 for $|x|\geqslant a_n/n$

$$|P_n(x)W_Q(x)(|x|+(a_n/n)_r)^r|$$
$$\leqslant C(n/a_n)^{1/2}[\max\{n^{-2/3},1-(|x|/a_n)\}]^{1/4}A_n(x_{kn})/b_n\}^{1/2}|x-x_{kn}|. \tag{2.14}$$

If $|x|\leqslant \varepsilon a_n/n$, then by Theorem 1.2(i) we see that (2.14) is also true. Let $|x|\leqslant a_n(1+Ln^{-2/3})$, $L>0$. Then, using Theorem 1.4 we can choose x_{kn} such that

$$|x-x_{kn}| \leqslant C(a_n/n)[\max\{n^{-2/3},1-(|x|/a_n)\}]^{-1/2}.$$

Hence, by Lemma 2.11 we have

$$|P_n(x)W_Q(x)(|x|+(a_n/n)_r)^r|$$
$$\leqslant C(a_n/n)^{1/2}[\max\{n^{-2/3},1-(|x|/a_n)\}]^{-1/4}\{A_n(x_{kn})/b_n\}^{1/2}$$
$$\leqslant Ca_n^{-1/2}[\max\{n^{-2/3},1-(|x|/a_n)\}]^{-1/4}\chi_n^{1/(2\alpha)}.$$

Especially, since $a_nP_n^2(x)W_Q^2(x)(|x|+(a_n/n)_r)^{2r}\leqslant C\chi_n^{1/\alpha}$ for $|x|\leqslant a_n/2$, we see $\chi_n\leqslant C\chi_n^{1/\alpha}$. From $1<\alpha$ we have

$$X_n\leqslant C, \quad n\geqslant 1. \tag{2.15}$$

Consequently, for $|x|\leqslant a_n(1+Ln^{-2/3})$ we have

$$|P_n(x)W_Q(x)(|x|+(a_n/n)_r)^r|\leqslant Ca_n^{-1/2}[\max\{n^{-2/3},1-(|x|/a_n)\}]^{-1/4}. \qquad \square$$

Proof of Theorem 1.7. From Theorem 1.1 and Lemma 2.7 we have

$$\begin{aligned}1&=\int_{-\infty}^{\infty}P_n^2(t)W_{rQ}^2(t)\,dt\leqslant C\int_{|x|\leqslant a_n}P_n^2(t)W_{rQ}^2(t)\,dt\\&\leqslant C\left(\int_{|x|\leqslant\varepsilon a_n/n}P_n^2(t)W_{rQ}^2(t)\,dt+\int_{\varepsilon a_n/n\leqslant|x|\leqslant a_n}P_n^2(t)W_{rQ}^2(t)\,dt\right)\\&\leqslant C\int_{\varepsilon a_n/n\leqslant|x|\leqslant a_n}P_n^2(t)W_{rQ}^2(t)\,dt.\end{aligned}$$

Here, by Theorem 1.8

$$\int_{\varepsilon a_n/n\leqslant|x|\leqslant\varepsilon a_n}P_n^2(t)W_{rQ}^2(t)\,dt\leqslant C\int_{|x|\leqslant\varepsilon a_n}a_n^{-1}\,dt\leqslant C\varepsilon.$$

Therefore, for $\varepsilon>0$ small enough and n large enough there is $\delta>0$ such that

$$\int_{\varepsilon a_n\leqslant|t|\leqslant Da_n}P_n^2(t)W_{rQ}^2(t)\,dt\geqslant\delta \quad (D\geqslant 1).$$

Furthermore, for $|x|\leqslant Da_n$ and $\varepsilon a_n\leqslant|t|\leqslant Da_n$ we see

$$\bar{Q}(x,t)\leqslant Q''(Da_n)\sim n/a_n^2 \quad \text{(see (0.1) and (1.4))}.$$

On the other hand, $\bar{Q}(x,t)\geqslant CQ'(t)/t\geqslant Cn/a_n^2$, so $\bar{Q}(x,t)\sim n/a_n^2$. From this

$$A_n(x)/b_n\geqslant\int_{\varepsilon a_n\leqslant|t|\leqslant Da_n}P_n^2(t)W_{rQ}^2(t)\bar{Q}(x,t)\,dt\geqslant Cn/a_n^2.$$

Consequently, by Lemma 2.11 and (2.15) we have $A_n(x)\sim n/a_n$.

The second inequality of Theorem 1.7 follows the first inequality. In fact, using the Cauchy–Schwartz inequality, for $|x|\leqslant Da_n$

$$\begin{aligned}|B_n(x)|&\leqslant 2b_n\left\{\int_{-\infty}^{\infty}P_n^2(t)\bar{Q}(x,t)W_{rQ}^2(t)\,dt\right\}^{1/2}\left\{\int_{-\infty}^{\infty}P_{n-1}^2(t)\bar{Q}(x,t)W_{rQ}^2(t)\,dt\right\}^{1/2}\\&\leqslant Cn/a_n. \qquad \square\end{aligned}$$

Proof of Theorem 1.9.

(i) From (2.13), Theorems 1.7 and 1.2(i) we see $(n/a_n)^{2r+1} \sim nP_{n-1}^2(0)$ for $x_{kn} = 0$, that is,

$$|P_{n-1}(0)| \sim (n/a_n)^r a_n^{-1/2}.$$

Consequently, we have (1.6).

We show (1.7). Using (2.12) and (1.6), we see

$$(n/a_n)^{2r+1} \sim |a_n P_n'(0)(n/a_n)^r a_n^{-1/2}|.$$

Hence,

$$|P_n'(0)| \sim (n/a_n)^r n a_n^{-3/2}.$$

(ii) Let $x_{jn} \neq 0$. By (2.12) and (2.13)

$$\begin{aligned}
&|(d/dx)\{P_n(x)W_{rQ}(x)\}_{x=x_{jn}}|\\
&\quad = |A_n(x_{jn})P_{n-1}(x_{jn})W_{rQ}(x_{jn})|\\
&\quad = |[\lambda_n^{-1}(W_{rQ}^2; x_{kn})/\{b_n A_n(x_{jn})\}]^{1/2} A_n(x_{jn})W_{rQ}(x_{jn})|\\
&\quad = |\{\lambda_n^{-1}(W_{rQ}^2; x_{jn})W_{rQ}^2(x_{jn})\}^{1/2}\{A_n(x_{jn})/b_n\}^{1/2}|\\
&\quad \sim (n/a_n)^{1/2}[\max\{n^{-2/3}, 1-(|x_{jn}/a_n|)\}]^{1/4}(n^{1/2}/a_n)\\
&\quad \text{(by Theorems 1.2(iii), 1.7)}\\
&\quad = n a_n^{-3/2}[\max\{n^{-2/3}, 1-(|x_{jn}|/a_n)\}]^{1/4}.
\end{aligned}$$

Consequently, we have (1.8). Especially,

$$|(d/dx)\{P_n(x)W_{rQ}(x)\}_{x=x_{1n}}| \sim n a_n^{-3/2} n^{-1/6} \quad \text{(by Theorem 1.3)}. \qquad \square$$

Using Theorem 1.9, we can improve Lemmas 2.8 and 2.9.

Lemma 2.12. *There is a constant C such that $Ca_n/n \leqslant x_{[n/2],n}$ for every n. Here Lemma 2.9 is correct for all $x_{kn} \neq 0$.*

Proof. From (2.13) and Theorem 1.7 we see

$$\lambda_n^{-1}(W_{rQ}^2; x_{[n/2],n}) \sim nP_{n-1}^2(x_{[n/2],n}),$$

and by Theorem 1.2 we have

$$\lambda_n(W_{rQ}^2; x_{[n/2],n}) \sim (a_n/n)^{2r+1}.$$

Therefore, $P_{n-1}(x_{[n/2],n}) \sim (n/a_n)^r a_n^{-1/2}$. If we set

$$x_{[n/2],n} = \varepsilon_n(a_n/n), \quad \varepsilon_n \to 0,$$

then we see

$$P_{n-1}(x_{[n/2],n})/x_{[n/2],n} \sim \{(n/a_n)^{r+1} a_n^{-1/2}\}/\varepsilon_n.$$

Hence, there is ξ_n, $0 < \xi_n < x_{[n/2],n}$, such that

$$P'_{n-1}(\xi_n) \sim \{(n/a_n)^{r+1} a_n^{-1/2}\}/\varepsilon_n.$$

However, this contradicts $P'_{n-1}(0) \sim (n/a_n)^{r+1} a_n^{-1/2}$ in Theorem 1.9 (we note the concavity of $y = |P_{n-1}(x)|$). For Lemma 2.9 it is trivial. □

Proof of Theorem 1.10. By (2.14) and Theorem 1.7 we have

$$\begin{aligned} &|P_n(x) W_Q(x)(|x| + (a_n/n)_r)^r| \\ &\leqslant C(n/a_n)^{1/2}[\max\{n^{-2/3}, 1 - (|x|/a_n)\}]^{1/4}\{A_n(x_{jn})/b_n\}^{1/2}|x - x_{jn}| \\ &\leqslant Cna_n^{-3/2}[\max\{n^{-2/3}, 1 - (|x|/a_n)\}]^{1/4}|x - x_{jn}|. \end{aligned} \tag{2.16}$$

Using (1.3) and Theorem 1.6 to $n+1$, we see the following:

$$\lambda_{n+1}^{-1}(W_{rQ}^2; x_{k,n+1}) = b_{n+1} P'_{n+1}(x_{k,n+1}) P_n(x_{k,n+1}),$$

$$P'_{n+1}(x_{k,n+1}) = A_{n+1}(x_{k,n+1}) P_n(x_{k,n+1}).$$

Therefore, from Theorem 1.7

$$P_n^2(x_{k,n+1}) \sim (1/n) \lambda_{n+1}^{-1}(W_{rQ}^2; x_{k,n+1}), \tag{2.17}$$

hence, using Theorem 1.2(iii), for $x_{j,n+1} \neq 0$ we see

$$|P_n(x_{j,n+1}) W_{rQ}(x_{j,n+1})| \sim a_n^{-1/2}[\max\{n^{-2/3}, 1 - (|x_{j,n+1}|/a_n)\}]^{1/4}. \tag{2.18}$$

Now, we use formula (2.16) for $x = x_{j,n+1}$, then by (2.18) we see $Ca_n/n \leqslant |x_{j+1,n} - x_{jn}| \leqslant |x_{j-1,n} - x_{jn}|$. If $x_{j,n+1} = 0$, then by Lemma 2.12 we have $Ca_n/n \leqslant |x_{jn}| = (1/2)|x_{j-1,n} - x_{jn}|$. Consequently, the first formula of Theorem 1.10 is proved. The second formula follows from Theorem 1.4. □

Proof of Theorem 1.11. (i) Let n be even. By (1.6),

$$\max_{|x| \leqslant x_{[n/2]n}} |P_n(x)| = |P_n(0)| \sim (n/a_n)^r a_n^{-1/2}.$$

If n is odd, then by (2.17), Theorems 1.2(i) and 1.10

$$|P_n(x_{[(n+1)/2],n+1})| \sim (n/a_n)^r a_n^{-1/2}.$$

Therefore, we have (1.9). In other cases, (1.10) follows from Theorem 1.8 and (2.18).

(ii) First, we show (1.12). By (1.8),

$$\max_{x_{kn} \leqslant x \leqslant x_{k-1,n}} |P'_n(x) W_{rQ}(x)| \geqslant Cna_n^{-3/2}.$$

By Theorem 1.6 we see

$$P'_n(x) = A_n(x) P_{n-1}(x) - B_n(x) P_n(x) - 2r\{P_n(x)/x\}^*,$$

hence,

$$\max_{x_{kn}\leqslant x\leqslant x_{k-1,n}} |P_n'(x)W_{rQ}(x)| \leqslant C \max_{x_{kn}\leqslant x\leqslant x_{k-1,n}} [|A_n(x)P_{n-1}(x)W_{rQ}(x)| + |B_n(x)P_n(x)W_{rQ}(x)| + 2|r|\,|\{P_n(x)/x\}^* W_{rQ}(x)|]. \tag{2.19}$$

Here we use Theorems 1.8 and 1.7. For $Ca_n/n\leqslant|x|$ we see that from (2.19)

$$\max_{x_{kn}\leqslant x\leqslant x_{k-1,n}} |P_n'(x)W_{rQ}(x)|\leqslant Cna_n^{-3/2}.$$

By (1.8) we conclude (1.12).

We show (1.11). Let n be odd, and let

$$\max_{|x|\leqslant x_{[n/2],n}} |P_n(x)| = |P_n(\bar{x})|, \quad 0<\bar{x}<x_{[n/2],n}.$$

Since $y=|P_n(x)|$ is concave on $[0,\bar{x}]$, we see

$$\max_{|x|\leqslant\bar{x}} |P_n'(x)| = |P_n'(0)|\sim(n/a_n)^r na_n^{-3/2} \quad \text{(by (1.7))}. \tag{2.20}$$

Here we set $\bar{x}=\varepsilon_n a_n/n$. If $\varepsilon_n\to 0$, then by (1.9)

$$|P_n(\bar{x})/\bar{x}|\sim(1/\varepsilon_n)(n/a_n)^r na_n^{-3/2}.$$

This contradicts (2.20). So we see $\bar{x}\sim a_n/n$. If in (2.19) we exchange x_{kn} for $\bar{x}$, and set $x_{k-1,n}=x_{[n/2],n}$, then the consideration under the inequality (2.19) is correct similarly. So we have

$$\max_{\bar{x}\leqslant x\leqslant x_{[n/2],n}} |P_n'(x)W_{rQ}(x)|\leqslant Cna_n^{-3/2}. \tag{2.21}$$

Since we see $W_{rQ}(x)\sim(a_n/n)^r$ for $\bar{x}\leqslant x\leqslant x_{[n/2],n}$, inequality (2.21) implies

$$\max_{\bar{x}\leqslant x\leqslant x_{[n/2],n}} |P_n'(x)|\leqslant C(n/a_n)^r na_n^{-3/2}. \tag{2.22}$$

Consequently, by (2.20) and (2.22) we obtain (1.11).

Let n be even. From $W_{rQ}(x_{[n/2],n})\sim(a_n/n)^r$ and (1.8) we have

$$|P_n'(x_{[n/2],n})|\sim(n/a_n)^r na_n^{-3/2}. \tag{2.23}$$

By Theorem 1.6, $P_n'(x)=A_n(x)P_{n-1}(x)-B_n(x)P_n(x)$. Using Theorem 1.7 and (1.9), we see

$$\max_{0\leqslant x\leqslant x_{[n/2],n}} |P_n'(x)|\leqslant C(n/a_n)^r na_n^{-3/2}.$$

So by the symmetry of $P_n'(x)$ and (2.23) we get (1.11). □

For an application we need an exact result of Theorem 1.11. We estimate the values in the neighborhood of x_{kn}, and otherwise.

Proof of Corollary 1.12. Let $x_{kn} \neq 0$. For the interval $[x_{kn}, x_{k-1,n}]$ we set

$$\max_{x_{kn} \leqslant x \leqslant x_{k-1,n}} |P_n(x) W_{rQ}(x)| = |P_n(\bar{x}_{kn}) W_{rQ}(\bar{x}_{kn})|, \quad x_{kn} < \bar{x}_{kn} < x_{k-1,n}.$$

We consider the points $A(x_{kn}, 0)$, $B(\bar{x}_{kn}, 0)$ and $C(\bar{x}_{kn}, |P_n(\bar{x}_{kn}) W_{rQ}(\bar{x}_{kn})|)$.

(a) Let $y = |P_n(x) W_{rQ}(x)|$ be concave on $[x_{kn}, \bar{x}_{kn}]$. We denote the tangent of $y = |P_n(x) W_{rQ}(x)|$ at A and C by l and l', respectively. Let l and l' intersect at D, and let us denote the middle point of the segment AD by P. Furthermore, let us denote the line PC and the graph of $y = |P_n(x) W_{rQ}(x)|$ intersect by Q, and the x-coordinates of D, P and Q by $x = d$, $x = p$ and $x = q$, respectively. We also consider the line AC.

(b) Let $y = |P_n(x) W_{rQ}(x)|$ be convex–concave on $[x_{kn}, \bar{x}_{kn}]$. For the graph of $y = |P_n(x) W_{rQ}(x)|$ we consider the tangent l'' passing through the point A, where the curve is situated under l'' on $[x_{kn}, \bar{x}_{kn}]$. The other notations in (a) are defined similarly. We denote the tangent of $y = |P_n(x) W_{rQ}(x)|$ at A by l. Let $E(\neq A)$ be the point where the tangent l intersects the graph $y = |P_n(x) W_{rQ}(x)|$ again, and let us denote the x-coordinate of E by e. The line AE^* expresses the line AE if $\bar{x}_{kn} \leqslant e$, or the line AC if $e < \bar{x}_{kn}$ ($E^* = E$ or C).

(c) From (a) or (b) we obtain the following: Let $|x| \leqslant \eta a_n,\ 0 < \eta < 1$.

(1) We see that on $[x_{kn}, \bar{x}_{kn}]$ the graph of $y = |P_n(x) W_{rQ}(x)|$ is situated between the lines AD and AC, or between the lines AD and AE^*.

(2) The x-coordinate d of D satisfies $d - x_{kn} \sim a_n/n$, hence, $p - x_{kn} \sim a_n/n$ and $q - x_{kn} \sim a_n/n$.

(3) The slope m of the line PC satisfies $|m| < |\{P_n(x) W_{rQ}(x)\}'|$ for $x_{kn} \leqslant x \leqslant q$.

The proof of (c) follows from (a), (b), especially, $d - x_{kn} \sim a_n/n$ is shown as follows. If $d - x_{kn} = \varepsilon_n a_n/n$, $\varepsilon_n \to 0$, then the slope of the line AD exceeds largely over the value given in Theorem 1.11. So it is contradictory. From (3) we see

$$|P_n'(x) W_{rQ}(x) + \{(r/x) - Q'(x)\} P_n(x) W_{rQ}(x)| \sim n a_n^{-3/2}, \quad x_{kn} \leqslant x \leqslant q,$$

so there exists a constant δ' such that

$$|P_n'(x) W_{rQ}(x)| \sim n a_n^{-3/2}, \quad x_{kn} \leqslant x \leqslant x_{kn} + \delta' a_n/n.$$

Let $x_{kn} = 0$. Then we treat a graph of $y = |P_n(x)|$ instead of $y = |P_n(x) W_{rQ}(x)|$ in the above consideration. So we obtain the same results described above with respect to $y = |P_n(x)|$. Now, from these the proof of Corollary 1.12 follows. □

Proof of Theorem 1.13. By Theorem 1.1 we see

$$\begin{aligned}\zeta_n^4 &= \sup_{x \in \mathbf{R}} |P_n(x)|^4 W_Q^4(x) (|x| + (a_n)_r)^{4r} |1 - (|x|/a_n)| \\ &\leqslant C \sup_{|x| \leqslant a_n} |P_n(x)|^4 W_Q^4(x) (|x| + (a_n)_r)^{4r} \{1 - (|x|/a_n)^2\}\end{aligned}$$

(for $W_{rQ}^4(x)$ and $P\in\prod_{4n}$ we can take the interval $[-a_n, a_n]$)

$$\leqslant C(a_n^{-1/2})^4 \text{ (by Theorem 1.8 and (2.3))}.$$

Therefore, we have $\zeta_n \leqslant Ca_n^{-1/2}$.

We show the inverse inequality. By Theorem 1.1

$$1 = \int_{-\infty}^{\infty} P_n^2(x)W_{rQ}^2(x)\,dx \leqslant C\int_{|x|\leqslant a_n} P_n^2(x)W_{rQ}^2(x)\,dx$$

$$\leqslant C\zeta_n^2 \int_{|t|\leqslant a_n} \{1-(|t|/a_n)\}^{-1/2}\left(\frac{|t|}{|t|+(a_n/n)_r}\right)^{2r} dt \quad \text{(by the definition of } \zeta_n)$$

$$\leqslant C\zeta_n^2 a_n \int_0^1 (1-s)^{-1/2}\left(\frac{s}{s+(a_n/n)_r/a_n}\right)^{2r} ds \leqslant C\zeta_n^2 a_n.$$

Consequently, we see $\zeta_n \sim a_n^{-1/2}$. □

To prove Theorem 1.14 we need the following lemma.

Lemma 2.13 (Levin and Lubinsky [LL1, Theorem 1.9]). *Let $Q(0)=0$. For $P\in\prod_n$ we have*

$$|(PW_Q)'(x)| \leqslant C(n/a_n)[\max\{n^{-2/3}, 1-(|x|/a_n)\}]^{1/2}\|PW_Q\|_{C_\infty(\mathbf{R})}.$$

Proof of Theorem 1.14. By Theorem 1.8 we see for $|x|\leqslant a_n(1-Ln^{-2/3})$

$$|P_n(x)W_Q(x)(|x|+(a_n/n)_r)^r| \leqslant Ca_n^{-1/2}|1-(|x|/a_n)|^{-1/4} \leqslant Ca_n^{-1/2}n^{1/6}.$$

Therefore, by Theorem 1.1

$$\|P_nW_Q(|x|+(a_n/n)_r)^r\|_{C_\infty(\mathbf{R})} \leqslant Ca_n^{-1/2}n^{1/6}. \tag{2.24}$$

We show an inverse inequality. Let $|(|x|/a_n)-1| \leqslant Cn^{-2/3}$. Applying Lemma 2.13 to

$$(P_nW_{rQ})'(x) = x^r(P_nW_Q)'(x) + rx^{r-1}(P_nW_Q)(x),$$

we see

$$\begin{aligned}
&|(P_nW_{rQ})'(x)| \\
&\quad \leqslant C[x^r(n/a_n)n^{-1/3}\|P_nW_Q\|_{L_\infty(|x|\leqslant 2a_n)} + x^r(1/a_n)\|P_nW_Q\|_{L_\infty(|x|\leqslant 2a_n)}] \\
&\quad \leqslant C[(n/a_n)n^{-1/3} + (1/a_n)]\|P_nW_{rQ}\|_{L_\infty(|x|\leqslant 2a_n)} \\
&\quad \leqslant C(n/a_n)n^{-1/3}\|P_nW_{rQ}\|_{L_\infty(|x|\leqslant 2a_n)}.
\end{aligned}$$

If we set $x = x_{1n}$, then from (1.8),

$$na_n^{-3/2}n^{-1/6} \leqslant C(n/a_n)n^{-1/3}\|P_nW_{rQ}\|_{L_\infty(|x|\leqslant 2a_n)}.$$

From this we see

$$Ca_n^{-1/2}n^{1/6} \leqslant \|P_nW_{rQ}\|_{C_\infty(\mathbf{R})}.$$

Consequently, with (2.24) we obtain

$$\sup_{x\in\mathbf{R}} |P_n(x)W_Q(x)(|x|+(a_n/n)_r)^r| \sim a_n^{-1/2}n^{1/6}. \qquad \square$$

Proof of Theorem 1.15. Let $r \geqslant 0$. From Theorem 1.1 we see

$$||P_n' W_{rQ}||_{C_\infty(\mathbf{R})} \leqslant C||P_n' W_{rQ}||_{L_\infty(|x|\leqslant a_n(1-Kn^{-2/3}))}.$$

By Theorem 1.6

$$\begin{aligned} &P_n'(x)W_{rQ}(x) + 2r\{P_n(x)/x\}^* W_{rQ}(x) \\ &\quad = A_n(x)P_{n-1}(x)W_{rQ}(x) + B_n(x)P_n(x)W_{rQ}(x). \end{aligned}$$

Let n be odd, then we see that the polynomial $P_n(x)$ is odd. We define x^* as $\sup_{0\leqslant x\leqslant x_{[n/2],n}} |P_n(x)| = |P_n(x^*)|$, $0 < x^* < x_{[n/2],n}$. Since we see $\mathrm{sign}[P_n'(x)] = \mathrm{sign}[P_n(x)/x]$ in $[0, x^*]$, we have

$$\begin{aligned} |\{P_n(x)/x\}^* W_{rQ}(x)| &\leqslant |A_n(x)P_{n-1}(x)W_{rQ}(x)| + |B_n(x)P_n(x)W_{rQ}(x)| \\ &\leqslant Cna_n^{-3/2}, \quad 0 \leqslant x \leqslant x^*. \end{aligned}$$

For $x \in [x^*, x_{[n/2],n}]$ we see

$$\begin{aligned} \sup_{x^*\leqslant x\leqslant x_{[n/2],n}} |P_n(x)/x| &\leqslant \sup_{0\leqslant x\leqslant x^*} |P_n(x)/x| \\ &\leqslant Cna_n^{-3/2}. \end{aligned}$$

We use again Theorem 1.6.

$$\begin{aligned} |P_n'(x)W_{rQ}(x)| \leqslant [&|A_n(x)P_{n-1}(x)W_{rQ}(x)| + |B_n(x)P_n(x)W_{rQ}(x)| \\ &+ 2r|\{P_n(x)/x\}^* W_{rQ}(x)|]. \end{aligned} \tag{2.25}$$

For $x_{[n/2],n} \leqslant |x| \leqslant \eta a_n$, $0 < \eta < 1$, we have

$$|\{P_n(x)/x\}^* W_{rQ}(x)| \leqslant C(n/a_n)a_n^{-1/2}.$$

Let $\eta a_n \leqslant |x|$. By Theorem 1.14 we see

$$|\{P_n(x)/x\}^* W_{rQ}(x)| \leqslant Ca_n^{-3/2}n^{1/6}.$$

So we get

$$|\{P_n(x)/x\}^* W_{rQ}(x)| \leqslant C\{(n/a_n)a_n^{-1/2} + a_n^{-3/2}n^{1/6}\}, \quad x \in \mathbf{R}.$$

Therefore, from Theorems 1.7 and 1.8 we see that for $|x| \leqslant a_n$ inequality (2.25) implies $|P_n'(x)W_{rQ}(x)| \leqslant Cna_n^{-3/2}n^{1/6}$. From this we have

$$||P_n' W_{rQ}||_{C_\infty(\mathbf{R})} \leqslant Cna_n^{-3/2}n^{1/6}. \tag{2.26}$$

On the other hand, we see

$$\{P_n(x)W_{rQ}(x)\}' = P_n(x)'W_{rQ}(x) + rP_n(x)x^{r-1}W_Q(x) \\ - Q'(x)P_n(x)W_{rQ}(x).$$

If we set

$$\max_{|x|\leqslant a_n(1+Ln^{-2/3})} |P_n(x)W_{rQ}(x)| = |P_n(\bar{x})W_{rQ}(\bar{x})|,$$

then from $\{P_n(x)W_{rQ}(x)\}'_{x=\bar{x}} = 0$ we have

$$|P_n'(\bar{x})W_{rQ}(\bar{x})| = |Q'(\bar{x})P_n(\bar{x})W_{rQ}(\bar{x}) - r\{P_n(\bar{x})W_{rQ}(\bar{x})\}/\bar{x}| \\ \sim na_n^{-3/2}n^{1/6} \quad \text{(by Theorem 1.14)}.$$

Consequently by (2.26) we obtain

$$\max_{x\in\mathbf{R}} |P_n'(x)W_{rQ}(x)| \sim na_n^{-3/2}n^{1/6}. \qquad \square$$

3. Further properties of orthonormal polynomials

To get further properties of orthonormal polynomials we need to strengthen the conditions for the Freud exponential $Q(x)$. Let $v = 1, 2, 3, \ldots$. If $v = 1$, then we assume (0.1), and for $v\geqslant 2$ we suppose (0.1) and further that $Q\in C^{(v+1)}(\mathbf{R})$ and

$$0\leqslant xQ^{(j+1)}(x)/Q^{(j)}(x)\leqslant \tilde{B}, \quad j = 2, 3, \ldots, v,$$
$$Q^{(v+1)}(x)\uparrow \text{ (nondecreasing)}, \quad x\in(0,\infty), \tag{3.1}$$

where $\tilde{B}$ is a positive constant. For this $Q(x)$ the Freud-type weights $W_{rQ}(x)$ are defined by (0.3), and then we say that the weight $W_{rQ}(x)$ satisfies the condition $C(v)$. We consider the series of orthonormal polynomials $\{P_n(W_{rQ}^2; x)\}_{n=0}^{\infty}$ with weights (0.3). The polynomials $\{P_n(W_{rQ}^2; x)\}_{n=0}^{\infty}$ are constructed by (0.4) with $W_{rQ}(x)$.

When $v = 1$, in previous section we have obtained some properties of the orthonormal polynomials $\{P_n(W_{rQ}^2; x)\}_{n=0}^{\infty}$. In this section we investigate further properties of $\{P_n(W_{rQ}^2; x)\}_{n=0}^{\infty}$.

Our result analogies [KS]. We begin to estimate jth differential of $A_n(x)$ or $B_n(x)$ which is defined in Theorem 1.6.

Theorem 3.1. *Let Q satisfy the condition $C(v)$, then for $|x|\leqslant Da_n$, $D>0$,*

$$|A_n^{(j)}(x)|\leqslant Cn/a_n^{j+1}, \quad |B_n^{(j)}(x)|\leqslant Cn/a_n^{j+1}, \quad j = 0, 1, \ldots, v-1.$$

We need an extension of Theorem 3.1.

Theorem 3.2. *Let Q satisfy the condition $C(v+1)$. For $|x| \leqslant Da_n,\ D>0$, we have the following estimates*:

(i) *For each odd integer $j,\ 1 \leqslant j \leqslant v-1$, we have*

$$|A_n^{(j)}(x)| \leqslant C|x|n/a_n^{j+2}.$$

(ii) *For each even integer $j,\ 0 \leqslant j \leqslant v-1$, we have*

$$|B_n^{(j)}(x)| \leqslant C|x|n/a_n^{j+2}.$$

Theorem 3.3. *We have the following differential equation*:

(i) *For any odd integer $n \geqslant 1$*

$$\begin{aligned} &P_n'' - (Q' + A_n'/A_n)P_n' \\ &\quad + \{(b_n A_n A_{n-1}/b_{n-1}) + B_n B_{n-1} - (xA_{n-1}B_n/b_{n-1}) \\ &\quad + B_n' - (A_n' B_n/A_n) - 2r(A_{n-1}/b_{n-1})\}P_n \\ &\quad + 2r(xP_n' - P_n)/x^2 + 2r(B_{n-1} - A_n'/A_n)(P_n/x) = 0. \end{aligned}$$

(ii) *For any even integer $n \geqslant 2$*

$$\begin{aligned} &P_n'' - (Q' + A_n'/A_n)P_n' \\ &\quad + \{(b_n A_n A_{n-1}/b_{n-1}) + B_n B_{n-1} - (xA_{n-1}B_n/b_{n-1}) \\ &\quad + B_n' - (A_n' B_n/A_n)\}P_n \\ &\quad + 2r(P_n'/x) + 2rB_n(P_n/x) = 0. \end{aligned}$$

We rewrite Theorem 3.3 as follows.

(i) For any odd integer n

$$a(x)P_n''(x) + b(x)P_n'(x) + c(x)P_n(x) + D(x) + E(x) = 0, \tag{3.2}$$

where

$$a(x) = A_n(x), \quad b(x) = -Q'(x)A_n(x) - A_n'(x),$$

$$\begin{aligned} c(x) &= \{b_n A_n^2(x)A_{n-1}(x)/b_{n-1}\} + A_n(x)B_n(x)B_{n-1}(x) \\ &\qquad - \{xA_n(x)A_{n-1}(x)B_n(x)/b_{n-1}\} + A_n(x)B_n'(x) - A_n'(x)B_n(x) \\ &\qquad - 2r\{A_n(x)A_{n-1}(x)/b_{n-1}\} \\ &= c_1(x) + c_2(x) + c_3(x) + c_4(x) + c_5(x) + c_6(x), \end{aligned}$$

$$D(x) = 2r\{A_n(x)B_{n-1}(x) - A_n'(x)\}\{P_n(x)/x\},$$

$$E(x) = 2rA_n(x)[\{xP_n'(x) - P_n(x)\}/x^2]. \tag{3.3}$$

(ii) For any even integer n

$$a(x)P_n''(x) + b(x)P_n'(x) + c(x)P_n(x) + D(x) + E(x) = 0, \tag{3.4}$$

where

$$a(x) = A_n(x), \quad b(x) = -Q'(x)A_n(x) - A_n'(x),$$

$$\begin{aligned} c(x) &= \{b_n A_n^2(x)A_{n-1}(x)/b_{n-1}\} + A_n(x)B_n(x)B_{n-1}(x) \\ &\quad - \{xA_n(x)A_{n-1}(x)B_n(x)/b_{n-1}\} + A_n(x)B_n'(x) - A_n'(x)B_n(x) \\ &= c_1(x) + c_2(x) + c_3(x) + c_4(x) + c_5(x), \end{aligned}$$

$$D(x) = 2rA_n(x)B_n(x)\{P_n(x)/x\}, \quad E(x) = A_n(x)\{P_n'(x)/x\}. \tag{3.5}$$

By (3.2) and (3.4) for $j = 0, 1, \dots, v-2$ $(v \geqslant 2)$ we consider the following differential equations:

$$\begin{aligned} &a(x)P_n''(x) + b(x)P_n'(x) + c(x)P_n(x) + D(x) + E(x) = 0, \quad j = 0, \\ &a(x)P_n'''(x) + \{a'(x) + b(x)\}P_n''(x) + \{b'(x) + c(x)\}P_n'(x) \\ &\quad + c'(x)P_n(x) + D'(x) + E'(x) = 0, \quad j = 1, \\ &a(x)P_n^{(j+2)}(x) + \{ja'(x) + b(x)\}P_n^{(j+1)}(x) \\ &\quad + \sum_{s=0}^{j-2} \left\{ \binom{j}{s+2} a^{(s+2)}(x) + \binom{j}{s+1} b^{(s+1)}(x) + \binom{j}{s} c^{(s)}(x) \right\} P_n^{(j-s)}(x) \\ &\quad + \{b^{(j)}(x) + jc^{(j-1)}(x)\}P_n'(x) + c^{(j)}(x)P_n(x) + D^{(j)}(x) + E^{(j)}(x) = 0, \\ &\quad j = 2, 3, \dots, v-2. \end{aligned}$$

Simply we write

$$\begin{aligned} &A_2^{[0]}(x)P_n''(x) + A_1^{[0]}(x)P_n'(x) + A_0^{[0]}(x)P_n(x) + D^{[0]}(x) + E^{[0]}(x) = 0, \quad j = 0, \\ &A_3^{[1]}(x)P_n'''(x) + A_2^{[1]}(x)P_n''(x) + A_1^{[1]}(x)P_n'(x) \\ &\quad + A_0^{[1]}(x)P_n(x) + D^{[1]}(x) + E^{[1]}(x) = 0, \quad j = 1, \\ &A_{j+2}^{[j]}(x)P_n^{(j+2)}(x) + A_{j+1}^{[j]}(x)P_n^{(j+1)}(x) + \sum_{s=0}^{j} A_{j-s}^{[j]}(x)P_n^{(j-s)}(x) \\ &\quad + D^{[j]}(x) + E^{[j]}(x) = 0, \quad j = 2, 3, \dots, v-2. \end{aligned} \tag{3.6}$$

We define

$$\langle i \rangle = \begin{cases} 1 & (i\text{: odd}), \\ 0 & (i\text{: even}), \end{cases} M_n(Q; x) = |x|/a_n^2 + |Q'(x)|.$$

Theorem 3.4. *Let* $v \geqslant 2$, *and let* Q *satisfy the condition* $C(v+1)$. *Then for* $|x| \leqslant Da_n,\ D>0$, *and* $j = 0, 1, \ldots, v-2$ *we have the following estimates*:

$$A_{j+2}^{[j]}(x) \sim (n/a_n), \quad |A_{j+1}^{[j]}(x)| \leqslant CM_n(Q;x)(n/a_n),$$
$$|A_{j-s}^{[j]}(x)| \leqslant C|x|^{\langle s \rangle}(n^3/a_n^{s+3+\langle s \rangle}), \quad s = 0, 1, \ldots, j,$$

where the constant C *is independent of* $n,\ x$.

Eqs. (3.6) are rewritten as follows.

Theorem 3.5. *Let* $v \geqslant 2$, *and let* Q *satisfy the condition* $C(v+1)$. *Then for* $j = 0, 1, \ldots, v-2$ *we have the following equations*:

$$B_{j+2}^{[j]}(x)P_n^{(j+2)}(x) + B_{j+1}^{[j]}(x)P_n^{(j+1)}(x) + \sum_{s=0}^{j} B_{j-s}^{[j]}(x)P_n^{(j-s)}(x) = 0,$$

where for $x_{kn} \neq 0$ *we see*

$$|B_{j+2}^{[j]}(x_{kn})| = |A_{j+2}^{[j]}(x_{kn})| \sim n/a_n,$$
$$|B_{j+1}^{[j]}(x_{kn})| \leqslant C\{M_n(Q;x_{kn}) + 1/|x_{kn}|\}(n/a_n),$$
$$|B_{j-s}^{[j]}(x_{kn})| \leqslant C[\{|x_{kn}|^{\langle s \rangle}n^3/a_n^{s+3+\langle s \rangle}\} + \{(n/a_n)^{s+2}/|x_{kn}|\}],$$
$$s = 0, 1, \ldots, j.$$

For any odd integer n *and* $x_{kn} = 0$ *we have*

$$|B_{j+2}^{[j]}(0)| = \{1 + 2r/(j+2)\}|A_{j+2}^{[j]}(0)| \sim n/a_n, \quad |B_{j+1}^{[j]}(0)| \leqslant C(n/a_n)^2,$$

$$|B_{j-s}^{[j]}(0)| \leqslant C[\{0^{\langle s \rangle}n^3/a_n^{s+3+\langle s \rangle}\} + n^2/a_n^{s+3}] \leqslant C(n^3/a_n^{s+3}),$$
$$s = 0, 1, \ldots, j. \tag{3.7}$$

The following theorem is applicable.

Theorem 3.6. *Let* $v \geqslant 2$, *and let* Q *satisfy the condition* $C(v+1)$, *then for* $i = 1, 2, \ldots, v$ *and* $x_{kn} \neq 0$

$$|P_n^{(i)}(x_{kn})| \leqslant C\{M_n(Q;x_{kn}) + 1/|x_{kn}|\}^{1-\langle i \rangle}(n/a_n)^{i-2+\langle i \rangle}|P_n'(x_{kn})|.$$

For any odd integer n *and* $x_{kn} = 0$, *using* (3.7), *we have*

$$|P_n^{(i)}(0)| \leqslant C(n/a_n)^{i-1}|P_n'(0)|, \quad i = 1, 2, \ldots, v.$$

After this we prove the above theorems. To prove Theorem 3.1 we need the following lemma.

Lemma 3.7. *Let $j = 1, 2, \ldots, \nu - 1$, and let $K \geqslant 2$ be a constant. Then there exist $D \geqslant K + 1$, $d > 0$ such that for $|x| \leqslant Ka_n$*

$$J_n(x) = \int_{|t| \geqslant Da_n} \left[P_n^2(t) \left\{ \frac{j!}{(t-x)^{j+1}} \right\} \left\{ Q'(t) - \sum_{i=0}^{j} (1/i!) Q^{(i+1)}(x)(t-x)^i \right\} \right] \times W_{rQ}^2(t)\, dt \leqslant C \exp(-dn).$$

Proof. For each $i = 0, 1, \ldots, j$

$$|Q^{(i+1)}(x)(t-x)^i| \leqslant C|Q^{(i+1)}(t)(t-x)^i| \leqslant C|\{Q'(t)/t^i\}(t-x)^i| \leqslant C|Q'(t)| \leqslant C|Q'(t)|^2 \quad \text{(see (3.1))}.$$

Since $|t - x| \geqslant a_n$, we see

$$\left| \left\{ \frac{j!}{(t-x)^{j+1}} \right\} \left\{ Q'(t) - \sum_{i=0}^{j} \left(\frac{1}{i!} \right) Q^{(i+1)}(x)(t-x)^i \right\} \right| \leqslant Ca_n^{-(j+1)} |Q'(t)|^2.$$

Hence

$$J_n(x) \leqslant Ca_n^{-(j+1)} \int_{|t| \geqslant Da_n} P_n^2(t) \{Q'(t)\}^2 W_{rQ}^2(t)\, dt.$$

On the other hand, using the method of [Ba1, p. 221] for $|t| \geqslant (D/2)q_{2n}$, we see

$$P_n^2(t) W_{rQ}^2(t) \leqslant C2^{16n} (q_{2n}/|t|)^{2(n-r)} (1/q_{2n}) \int_{|t| \leqslant q_{2n}} P_n^2(t) W_{rQ}^2(t)\, dt \leqslant C2^{16n} q_{2n}^{2(n-r)-1} |t|^{-2(n-r)}.$$

Since $|Q'(t)|^2 \leqslant Ct^{2(B-1)}$ for a constant $B > 0$ [LL1, Lemma 5.1] and $q_{2n} < 2q_n < 2a_n$, for a constant $D > 0$ large enough we have

$$\begin{aligned} &\int_{|t| \geqslant Da_n} P_n^2(t) \{Q'(t)\}^2 W_{rQ}^2(t)\, dt \\ &\leqslant C2^{16n} q_{2n}^{2(n-r)-1} \int_{(D/2)q_{2n}}^{\infty} t^{-2(n-r)} \{Q'(t)\}^2\, dt \\ &\leqslant C2^{16n} q_{2n}^{2(n-r)-1} \int_{(D/2)q_{2n}}^{\infty} t^{-2n+2r+2B-2}\, dt \\ &\leqslant C(2^{18}/D^2)^n q_{2n}^{2B-2}/n \leqslant C \exp(-dn), \quad d > 0. \qquad \square \end{aligned}$$

Proof of Theorem 3.1. Let $Q \in C^{(j+2)}(\mathbf{R})$ and $|x| \leqslant Ka_n$. Then, for each $j = 1, 2, \ldots, \nu - 1$, $(\partial/\partial x)^j \bar{Q}(x, t)$ is continuous on the compact interval I with respect to $t \in I$ (hence bounded), and uniformly continuous for $x \in I$. Hence, using Lemma 3.7, there exists ξ ($t < \xi < x$ or $x < \xi < t$), and a constant $D > 0$ large enough such that

$$
\begin{aligned}
&|A_n^{(j)}(x)| \\
&\quad \leqslant Cb_n \left| \int_{|t|\leqslant Da_n} P_n^2(t) \left[\left\{ \frac{j!}{(t-x)^{j+1}} \right\} \right.\right. \\
&\qquad \left.\left. \times \left\{ Q'(t) - \sum_{i=0}^{j} \left(\frac{1}{i!}\right) Q^{(i+1)}(x)(t-x)^i \right\} \right] W_{rQ}^2(t)\, dt \right| \\
&\qquad + \exp(-dn) \quad (d>0) \ \ (x<\xi<t \text{ or } t<\xi<x) \\
&\quad \leqslant C\{1/(j+1)\} b_n \left| \int_{|t|\leqslant Da_n} P_n^2(t) Q^{(j+2)}(\xi) W_{rQ}^2(t)\, dt \right| + \exp(-dn) \\
&\quad \leqslant Cb_n |Q^{(j+2)}(Da_n)| \int_{-\infty}^{\infty} P_n^2(t) W_{rQ}^2(t)\, dt + \exp(-dn) \leqslant Cn/a_n^{j+1}.
\end{aligned}
$$

The estimate of $B_n^{(j)}(x)$ is obtained from that of $A_n^{(j)}(x)$ and the Cauchy–Schwarz inequality. □

If we strengthen the condition for Q, Theorem 3.1 is improved.

Proof of Theorem 3.2. We see that $A_n(x)$ is an even function. Using Theorem 3.1, for $Q \in C^{(v+2)}(\mathbf{R})$ we have $|A_n^{(j)}(x)| \leqslant Cn/a_n^{j+1}$, $j = 0, 1, \ldots, v$. Therefore, for positive odd integers $j \leqslant v-1$ there is ξ, $0<\xi<|x|$, such that

$$|A_n^{(j)}(x)| = |x|\{|A_n^{(j)}(x)|/|x|\} = |x||A_n^{(j+1)}(\xi)| \leqslant C|x|n/a_n^{j+2}.$$

Since $B_n(x)$ is an odd function, we can repeat the above method similarly. □

Proof of Theorem 3.3. If n is odd, then

$$P_n' = A_n P_{n-1} - B_n P_n - 2r(P_n/x), \tag{3.8}$$

$$P_n'' = A_n' P_{n-1} + A_n P_{n-1}' - B_n' P_n - B_n P_n' - 2r(xP_n' - P_n)/x^2. \tag{3.9}$$

The recurrence formula $xP_{n-1} = b_n P_n + b_{n-1} P_{n-2}$ means

$$P_{n-2} = (x/b_{n-1})P_{n-1} - (b_n/b_{n-1})P_n.$$

Using this (note even number $n-1$),

$$
\begin{aligned}
P_{n-1}' &= A_{n-1} P_{n-2} - B_{n-1} P_{n-1} \\
&= A_{n-1}\{(x/b_{n-1})P_{n-1} - (b_n/b_{n-1})P_n\} - B_{n-1}P_{n-1} \\
&= \{(xA_{n-1}/b_{n-1}) - B_{n-1}\}P_{n-1} - (b_n A_{n-1}/b_{n-1})P_n. \qquad (3.10)
\end{aligned}
$$

By (3.8),

$$P_{n-1} = (P_n'/A_n) + (B_n/A_n)P_n + 2rP_n/(xA_n), \tag{3.11}$$

and if we apply (3.11) to (3.10), then

$$\begin{aligned}
P'_{n-1} &= \{(xA_{n-1}/b_{n-1}) - B_{n-1}\} \\
&\quad \times \{(P'_n/A_n) + (B_n/A_n)P_n + 2rP_n/(xA_n)\} - (b_nA_{n-1}/b_{n-1})P_n \\
&= (1/A_n)\{(xA_{n-1}/b_{n-1}) - B_{n-1}\}P'_n + (1/A_n)\{(xA_{n-1}B_n/b_{n-1}) \\
&\quad - B_{n-1}B_n - (b_nA_{n-1}A_n/b_{n-1}) + 2r(A_{n-1}/b_{n-1})\}P_n \\
&\quad - 2r(B_{n-1}/A_n)(P_n/x). \qquad (3.12)
\end{aligned}$$

Applying (3.11) and (3.12) to (3.9),

$$\begin{aligned}
P''_n &= A'_n\{(1/A_n)P'_n + (B_n/A_n)P_n + 2r(1/A_n)(P_n/x)\} \\
&\quad + \{(xA_{n-1})/b_{n-1} - B_{n-1}\}P'_n \\
&\quad + \{(xA_{n-1}B_n/b_{n-1}) - B_{n-1}B_n - (b_nA_{n-1}A_n/b_{n-1}) + 2r(A_{n-1}/b_{n-1})\}P_n \\
&\quad - 2rB_{n-1}(P_n/x) - B'_nP_n - B_nP'_n - 2r\{(xP'_n - P_n)/x^2\} \\
&= -\{B_{n-1} + B_n - (xA_{n-1}/b_{n-1}) - A'_n/A_n\}P'_n \\
&\quad - \{(b_nA_{n-1}A_n/b_{n-1}) + B_{n-1}B_n - (xA_{n-1}B_n/b_{n-1}) \\
&\quad + B'_n - (A'_nB_n/A_n) - 2r(A_{n-1}/b_{n-1})\}P_n \\
&\quad - 2r\{(xP'_n - P_n)/x^2\} - 2r\{B_{n-1} - (A'_n/A_n)\}(P_n/x).
\end{aligned}$$

For any even integer n we can also show the result similarly. We have

$$\begin{aligned}
P''_n &= -\{B_{n-1} + B_n - (xA_{n-1}/b_{n-1}) - A'_n/A_n\}P'_n \\
&\quad - \{(b_nA_{n-1}A_n/b_{n-1}) + B_{n-1}B_n - (xA_{n-1}B_n/b_{n-1}) \\
&\quad + B'_n - (A'_nB_n/A_n)\}P_n - 2r(P'_n/x) - 2rB_n(P_n/x).
\end{aligned}$$

The equation $B_{n-1} + B_n - (xA_{n-1}/b_n) = -Q'$ (the coefficient of P'_n) is shown as follows. By the recurrence formula and $t/(t-x) = 1 + x/(t-x)$

$$\begin{aligned}
&B_n(x) + B_{n-1}(x) \\
&\quad = 2\int_{-\infty}^{\infty} P_{n-1}(t)\{b_nP_n(t) + b_{n-1}P_{n-2}(t)\}\bar{Q}(x,t)W_{rQ}^2(t)\,dt \\
&\quad = 2\int_{-\infty}^{\infty} P_{n-1}^2(t)\{Q'(t) - Q'(x)\}W_{rQ}^2(t)\,dt \\
&\qquad + 2x\int_{-\infty}^{\infty} P_{n-1}^2(t)\bar{Q}(x,t)W_{rQ}^2(t)\,dt
\end{aligned}$$

$$= 2\int_{-\infty}^{\infty} P_{n-1}^2(t)Q'(t)W_{rQ}^2(t)\,dt - Q'(x)$$
$$+ 2x\int_{-\infty}^{\infty} P_{n-1}^2(t)\bar{Q}(x,t)W_{rQ}^2(t)\,dt$$
$$= -Q'(x) + xA_{n-1}(x)/b_{n-1} \quad \text{(because } Q'(t) \text{ is an odd function)}.$$

From this, we have the result. □

Proof of Theorem 3.4. Let $j = 0, 1, \dots, v-2$, and $|x| \leqslant Ka_n$ $(K \geqslant 2)$. For $A_{j+2}^{|j|}(x) = A_n(x)$, the estimate follows from Theorem 1.7. By the definition in (3.6) and Theorem 3.2

$$|A_{j+1}^{|j|}(x)| \leqslant C\{|A_n'(x)| + |Q'(x)A_n(x)|\}$$
$$\leqslant C\{|x|n/a_n^3 + |Q'(x)|n/a_n\} \leqslant CM_n(Q;x)n/a_n.$$

For $A_{j-s}^{|j|}(x)$ we estimate $a^{(s+2)}(x)$, $b^{(s+1)}(x)$ and $c^{(s)}(x)$. We see $a^{(s+2)}(x) = A_n^{(s+2)}(x)$. The functions $a(x), b(x), c(x), c_1(x), \dots, c_6(x)$ are defined by (3.3) and (3.5). We use Theorem 3.2. Obviously,

$$|a^{(s+2)}(x)| = |A_n^{(s+2)}(x)| \leqslant C|x|^{\langle s \rangle} n/a_n^{s+3+\langle s \rangle}.$$

For $b^{(s+1)}(x)$, $s = -1, 0, \dots, j-1$, we see

$$b^{(s+1)}(x) = -\left\{A_n^{(s+2)}(x) + \sum_{p=0}^{s+1} \binom{s+1}{p} Q^{(p+1)}(x)A_n^{(s+1-p)}(x)\right\}.$$

We set

$$\sum = \sum_{p=0}^{s+1} \binom{s+1}{p} Q^{(p+1)}(x)A_n^{(s+1-p)}(x).$$

By (3.1) we see

$$|Q^{(p+1)}(x)| \leqslant Q^{(p+1)}(Ka_n) \leqslant CQ'(Ka_n)(Ka_n)^{-p} \leqslant Cna_n^{-(p+1)}.$$

From Theorem 3.2

$$|Q^{(p+1)}(x)A_n^{(s+1-p)}(x)| \leqslant Cn(Ka_n)^{-(p+1)}|x|^{\langle s+1-p \rangle} n/a_n^{s+2-p+\langle s+1-p \rangle}$$
$$\leqslant C|x|^{\langle s+1-p \rangle} n^2/a_n^{s+3+\langle s+1-p \rangle}.$$

Therefore, if $\langle s+1-p \rangle = \langle s \rangle$ or $0 = \langle s \rangle \neq \langle s+1-p \rangle$, then we see

$$|Q^{(p+1)}(x)A_n^{(s+1-p)}(x)| \leqslant C|x|^{\langle s \rangle} n^2/a_n^{s+3+\langle s \rangle}.$$

If $0 = \langle s+1-p \rangle \neq \langle s \rangle$, then $p+1$ is odd. By (3.1), the function $Q^{(p+1)}(x)/x$ is continuous and increasing on $(0, \infty)$. From these

$$|Q^{(p+1)}(x)A_n^{(s+1-p)}(x)| \leqslant C\{Q^{(p+1)}(Ka_n)/(Ka_n)\}|x|(n/a_n^{s+2-p})$$
$$\leqslant C|x|^{\langle s \rangle} n^2/a_n^{s+3+\langle s \rangle}.$$

Consequently,

$$|b^{(s+1)}(x)| \leqslant |A_n^{(s+2)}(x)| + \left|\sum\right| \leqslant C|x|^{\langle s \rangle} n^2/a_n^{s+3+\langle s \rangle}.$$

Next, we estimate $c^{(s)}(x)$. We prove only the case of odd number n, and omit the case of even n. Since $\{A_n^2(x)A_{n-1}(x)\}^{(s)}$ is a linear combination of $A_n^{(t)}(x)A_n^{(u)}(x)A_{n-1}^{(v)}(x)$, $t+u+v=s$, by Theorem 3.2, we have

$$|c_1^{(s)}(x)| \leqslant C \sum_{t,u,v,t+u+v=s} |x|^{\langle t \rangle + \langle u \rangle + \langle v \rangle} n^3/a_n^{s+3+\langle t \rangle + \langle u \rangle + \langle v \rangle}.$$

If s is even, then we see

$$|c_1^{(s)}(x)| \leqslant Cn^3/a_n^{s+3} = C|x|^{\langle s \rangle} n^3/a_n^{s+3+\langle s \rangle}.$$

If s is odd, by $\langle t \rangle + \langle u \rangle + \langle v \rangle \geqslant 1$ we see

$$|c_1^{(s)}(x)| \leqslant C|x|n^3/a_n^{s+4} \leqslant C|x|^{\langle s \rangle} n^3/a_n^{s+3+\langle s \rangle}.$$

For $c_2^{(s)}(x)$ by (3.3), (3.5) and Theorem 3.2

$$|c_2^{(s)}(x)| \leqslant C \sum_{t,u,v,t+u+v=s} |A_n^{(t)}(x)B_{n-1}^{(u)}(x)B_n^{(v)}(x)|$$
$$\leqslant C \sum_{t,u,v,t+u+v=s} |x|^{2+\langle t \rangle - \langle u \rangle - \langle v \rangle} n^3/a_n^{s+5+\langle t \rangle - \langle u \rangle - \langle v \rangle}.$$

If s is odd, by $1 + \langle t \rangle - \langle u \rangle - \langle v \rangle \geqslant 0$

$$|c_2^{(s)}(x)| \leqslant C|x|^{\langle s \rangle} n^3/a_n^{s+3+\langle s \rangle}.$$

If s is even, by $2 + \langle t \rangle - \langle u \rangle - \langle v \rangle \geqslant 0$

$$|c_2^{(s)}(x)| \leqslant C|x|^{\langle s \rangle} n^3/a_n^{s+3+\langle s \rangle}.$$

For $c_3^{(s)}(x)$ we consider

$$c_3^{(s)}(x) = (x/b_{n-1})\{A_n(x)A_{n-1}(x)B_n(x)\}^{(s)} + (s/b_{n-1})\{A_n(x)A_{n-1}(x)B_n(x)\}^{(s-1)} = c_{31}(x) + c_{32}(x), \quad \text{say}.$$

Here

$$|\{A_n(x)A_{n-1}(x)B_n(x)\}^{(s)}| \leqslant C \sum_{t,u,v,t+u+v=s} |x|^{1+\langle t \rangle + \langle u \rangle - \langle v \rangle} n^3/a_n^{s+4+\langle t \rangle + \langle u \rangle - \langle v \rangle} \leqslant Cn^3/a_n^{s+3}, \tag{3.13}$$

hence, we have $|c_{31}^{(s)}(x)| \leqslant C|x|^{\langle s \rangle} n^3/a_n^{s+3+\langle s \rangle}$. If in (3.13) we exchange s for $s-1$, then for any even s

$$(1/b_{n-1})|\{A_n(x)A_{n-1}(x)B_n(x)\}^{(s-1)}| \leqslant Cn^3/a_n^{s+3} = C|x|^{\langle s \rangle} n^3/a_n^{s+3+\langle s \rangle}.$$

Let s be odd. If we use the first inequality of (3.13) again by $1+\langle t\rangle+\langle u\rangle-\langle v\rangle\geqslant 1$

$$(1/b_{n-1})|\{A_n(x)A_{n-1}(x)B_n(x)\}^{(s-1)}|\leqslant C|x|^{\langle s\rangle}n^3/a_n^{s+3+\langle s\rangle}.$$

It is easy to see that $c_{32}(x)$ has the same estimate as $c_1^{(s)}(x)$.

We can estimate $c_4^{(s)}(x)$ as follows.

$$|c_4^{(s)}(x)|\leqslant C\sum_{t,u,t+u=s}|A_n^{(t)}(x)||B_n^{(u+1)}(x)|$$
$$\leqslant C\sum_{t,u,t+u=s}|x|^{1+\langle t\rangle-\langle u+1\rangle}n^2/a_n^{s+4+\langle t\rangle-\langle u+1\rangle}.$$

If s is even, then $|c_4^{(s)}(x)|\leqslant Cn^2/a_n^{s+3}$, and if s is odd, by $\langle t\rangle=\langle u+1\rangle$

$$|c_4^{(s)}(x)|\leqslant C|x|^{\langle s\rangle}n^2/a_n^{s+3+\langle s\rangle}. \tag{3.14}$$

For $c_5^{(s)}(x)$

$$|c_5^{(s)}(x)|\leqslant C\sum_{t,u,t+u=s}|A_n^{(t+1)}(x)|\,|B_n^{(u)}(x)|$$
$$\leqslant C\sum_{t,u,t+u=s}|x|^{1+\langle t+1\rangle-\langle u\rangle}n^2/a_n^{s+4+\langle t+1\rangle-\langle u\rangle},$$

so, we see that it has the same estimate as $c_4^{(s)}(x)$ in (3.14), that is

$$|c_5^{(s)}(x)|\leqslant C|x|^{\langle s\rangle}n^2/a_n^{s+3+\langle s\rangle}.$$

Finally, we estimate $c_6^{(s)}$ which comes out for any odd integer n.

$$|c_6^{(s)}|=|(1/b_{n-1})\{A_n(x)A_{n-1}(x)\}^{(s)}|\leqslant C\sum_{t,u,t+u=s}|A_n^{(t)}(x)|\,|A_{n-1}^{(u)}(x)|$$
$$\leqslant C\sum_{t,u,t+u=s}|x|^{1+\langle t\rangle+\langle u\rangle}n^2/a_n^{s+4+\langle t\rangle+\langle u\rangle}\leqslant C|x|^{\langle s\rangle}n^2/a_n^{s+3+\langle s\rangle}. \qquad \square$$

To prove Theorem 3.5 we need to consider the derivatives of $\{P_n(x)/x\}^*$.

Lemma 3.8. *Let $x_{kn}\neq 0$. We see*

$$\{P_n(x)/x\}^{(j)}_{x=x_{kn}}=(-1)^j x_{kn}^{-j}\sum_{i=1}^{j}(-1)^i(j!/i!)P_n^{(i)}(x_{kn})x_{kn}^{i-1},$$

$$[\{xP_n'(x)-P_n(x)\}/x^2]^{(j)}_{x=x_{kn}}=(-1)^{j+1}x_{kn}^{-(j+1)}\sum_{i=1}^{j+1}(-1)^i((j+1)!/i!)P_n^{(i)}(x_{kn})x_{kn}^{i-1}.$$

Let n be odd. For $x_{kn}=0$

$$\{P_n(x)/x\}^{(j)}_{x=x_{kn}}=p_n^{(j+1)}(0)/(j+1),$$

$$[\{xP_n'(x)-P_n(x)\}/x^2]^{(j)}_{x=0}=P_n^{(j+2)}(0)/(j+2),$$

and for $x_{kn} \neq 0$

$$\{P_n'(x)/x\}^{(j)}_{x=x_{kn}} = (-1)^j x_{kn}^{-j} \sum_{i=0}^{j-1} (-1)^i (j!/i!) P_n^{(i+1)}(x_{kn}) x_{kn}^{i-1}.$$

Proof. These follow easily from

$$P_n(x) = \sum_{i=1}^{n} \{P_n^{(i)}(x_{kn})/i!\}(x - x_{kn})^i,$$

$$P_n(x)/x = \sum_{i=1}^{n} \{P_n^{(i)}(x_{kn})/i!\}\{(x - x_{kn})^i/x\}. \qquad \square$$

Lemma 3.9. *Let* $x_{kn} \neq 0$. *We have*

$$|[\{A_n(x)B_{n-1}(x) - A_n'(x)\}\{P_n(x)/x\}]^{(j)}_{x=x_{kn}}| \leqslant C|x_{kn}^{-1}| \sum_{s=0}^{j-1} (n/a_n)^{s+2} |P_n^{(j-s)}(x_{kn})|,$$

$$|[A_n(x)\{(xP_n'(x) - P_n(x))/x^2\}]^{(j)}_{x=x_{kn}}| \leqslant C|x_{kn}^{-1}| \sum_{s=-1}^{j-1} (n/a_n)^{s+2} |P_n^{(j-s)}(x_{kn})|,$$

$$|[A_n(x)\{P_n'(x)/x\}]^{(j)}_{x=x_{kn}}| \leqslant C|x_{kn}^{-1}| \sum_{s=-1}^{j-1} (n/a_n)^{s+2} |P_n^{(j-s)}(x_{kn})|.$$

Especially, if n is odd, then for $x_{kn} = 0$ *we have*

$$|[\{A_n(x)B_{n-1}(x) - A_n'(x)\}\{P_n(x)/x\}]^{(j)}_{x=0}| \leqslant C \sum_{s=-1}^{j-1} (n^2/a_n^{s+3}) |P_n^{(j-s)}(0)|,$$

$$|[A_n(x)\{(xP_n'(x) - P_n(x))/x^2\}]^{(j)}_{x=0}| \leqslant C \sum_{s=-2}^{j-2} (n/a_n^{s+3}) |P_n^{(j-s)}(0)|.$$

Proof. We use the same method as we got the estimate $c_4(x)$ of (3.14). Using Theorem 3.1

$$|\{A_n(x)B_{n-1}(x) - A_n'(x)\}^{(i)}_{x=x_{kn}}| \leqslant C\{n^2/a_n^{i+2} + n/a_n^{i+2}\} \leqslant Cn^2/a_n^{i+2}.$$

Therefore, by Lemma 3.8 we see

$$\begin{aligned}
&|[\{A_n(x)B_{n-1}(x) - A_n'(x)\}\{P_n(x)/x\}]^{(j)}_{x=x_{kn}}| \\
&\quad = \left| \sum_{i=0}^{j} \binom{j}{i} \{A_n(x)B_{n-1}(x) - A_n'(x)\}^{(i)} \{P_n(x)/x\}^{(j-i)}_{x=x_{kn}} \right| \\
&\quad \leqslant C|x_{kn}|^{-1} \sum_{i=0}^{j-1} (n^2/a_n^{i+2}) \sum_{t=1}^{j-i} |P_n^{(t)}(x_{kn})||x_{kn}|^{-j+i+t}
\end{aligned}$$

$$\begin{aligned}
&\leqslant C|x_{kn}|^{-1}\sum_{t=1}^{j}|P_n^{(t)}(x_{kn})|\sum_{i=0}^{j-t}(n^2/a_n^{i+2})|x_{kn}|^{-j+i+t}\\
&\leqslant C|x_{kn}|^{-1}\sum_{t=1}^{j}|P_n^{(t)}(x_{kn})|\sum_{i=0}^{j-t}(n^2/a_n^{i+2})(n/a_n)^{j-i-t}\\
&\leqslant C|x_{kn}|^{-1}\sum_{t=1}^{j}(n/a_n)^{j-t+2}|P_n^{(t)}(x_{kn})|\\
&\leqslant C|x_{kn}|^{-1}\sum_{s=0}^{j-1}(n/a_n)^{s+2}|P_n^{(j-s)}(x_{kn})| \quad (\text{by } t=j-s),
\end{aligned}$$

$$\begin{aligned}
&|[A_n(x)\{(xP_n'(x)-P_n(x))/x^2\}]^{(j)}_{x=x_{kn}}|\\
&\quad=\left|\sum_{i=0}^{j}\binom{j}{i}A_n^{(i)}(x)\{(xP_n'(x)-P_n(x))/x^2\}^{(j-i)}_{x=x_{kn}}\right|\\
&\quad=\left|\sum_{i=0}^{j}\binom{j}{i}A_n^{(i)}(x)\{P_n(x)/x\}^{(j-i+1)}_{x=x_{kn}}\right|\\
&\quad=\left|\sum_{i=0}^{j}\binom{j}{i}\left[A_n^{(i)}(x)\sum_{t=0}^{j-i+1}\binom{j-i+1}{t}P_n^{(t)}(x)x^{-j+i+t-2}\right]_{x=x_{kn}}\right|\\
&\quad\leqslant C|x_{kn}|^{-1}\sum_{i=0}^{j}(n/a_n^{i+1})\sum_{t=0}^{j-i+1}|P_n^{(t)}(x_{kn})|\,|x_{kn}|^{-j+i+t-1}\\
&\quad\leqslant C|x_{kn}|^{-1}\sum_{t=1}^{j+1}|P_n^{(t)}(x_{kn})|\sum_{i=0}^{j-t+1}(n/a_n^{i+1})(n/a_n)^{j-i-t+1}\\
&\quad\leqslant C|x_{kn}|^{-1}\sum_{t=1}^{j+1}(n/a_n)^{j-t+2}|P_n^{(t)}(x_{kn})|\\
&\quad\leqslant C|x_{kn}|^{-1}\sum_{s=-1}^{j-1}(n/a_n)^{s+2}|P_n^{(j-s)}(x_{kn})| \quad (\text{by } t=j-s).
\end{aligned}$$

Furthermore,

$$\begin{aligned}
|[A_n(x)\{P_n'(x)/x\}]^{(j)}_{x=x_{kn}}| &= \left|\sum_{i=0}^{j}\binom{j}{i}A_n^{(i)}(x)\{P_n'(x)/x\}^{(j-i)}_{x=x_{kn}}\right|\\
&\leqslant C|x_{kn}|^{-1}\sum_{i=0}^{j}(n/a_n^{i+1})\sum_{t=0}^{j-i}|P_n^{(t+1)}(x_{kn})|\,|x_{kn}|^{-j+i+t}\\
&\leqslant C|x_{kn}|^{-1}\sum_{t=0}^{j}|P_n^{(t+1)}(x_{kn})|\sum_{i=0}^{j-t}(n/a_n^{i+1})(n/a_n)^{j-i-t}
\end{aligned}$$

$$\leqslant C|x_{kn}|^{-1} \sum_{t=0}^{j} (n/a_n)^{j-t+1} |P_n^{(t+1)}(x_{kn})|$$
$$\leqslant C|x_{kn}|^{-1} \sum_{s=-1}^{j-1} (n/a_n)^{s+2} |P_n^{(j-s)}(x_{kn})|$$
$$(\text{by } t+1 = j-s).$$

Consequently, we obtain the results.

Similarly, we have the case for $x_{kn} = 0$. □

Theorem 3.5 is shown as follows.

Proof of Theorem 3.5. Let $x_{kn} \neq 0$. By Theorem 3.4 and Lemma 3.9 for $j = 2, 3, \dots, v-2$, we have

$$|B_{j+2}^{[j]}(x_{kn})| = |A_{j+2}^{[j]}(x_{kn})| \sim n/a_n,$$

$$|B_{j+1}^{[j]}(x_{kn})| \leqslant C|A_{j+1}^{[j]}(x_{kn}) + (1/|x_{kn}|)(n/a_n)|$$
$$\leqslant C\{M_n(Q; x_{kn}) + 1/|x_{kn}|\}(n/a_n),$$

$$|B_{j-s}^{[j]}(x_{kn})| \leqslant C\{|A_{j-s}^{[j]}(x_{kn})| + (n/a_n)^{s+2}/|x_{kn}|\}$$
$$\leqslant C\{|x_{kn}|^{\langle s \rangle} n^3/a_n^{s+3+\langle s \rangle} + (n/a_n)^{s+2}/|x_{kn}|\}, \quad s = 0, 1, \dots, j.$$

Let n be odd. For $x_{kn} = 0$ we have (3.7)

$$|B_{j+2}^{[j]}(0)| = \{1 + 2r/(j+2)\}|A_{j+2}^{[j]}(0)| \sim n/a_n,$$
$$|B_{j+1}^{[j]}(0)| \leqslant C[|A_{j+1}^{[j]}(0)| + (n/a_n)^2] \leqslant C(n/a_n)^2,$$
$$|B_{j-s}^{[j]}(0)| \leqslant C\{|A_{j-s}^{[j]}(0)| + n^2/a_n^{s+3}\}$$
$$\leqslant C[\{0^{\langle s \rangle} n^3/a_n^{s+3+\langle s \rangle}\} + n^2/a_n^{s+3}], \quad s = 0, 1, \dots, j. \qquad \square \tag{3.15}$$

Proof of Theorem 3.6. Let $x_{kn} \neq 0$. If $i = 1, 2$, then the theorem is trivial. For $i = 2, 3, \dots, j+1$ $(j \leqslant v-2)$, we assume that the theorem is true. By Theorem 3.5 we see

$$|P_n^{(j+2)}(x_{kn})|$$
$$\leqslant |B_{j+1}^{[j]}(x_{kn})/B_{j+2}^{[j]}(x_{kn})||P_n^{(j+1)}(x_{kn})|$$
$$+ \sum_{s=0}^{j-1} |B_{j-s}^{[j]}(x_{kn})/B_{j+2}^{[j]}(x_{kn})||P_n^{(j-s)}(x_{kn})|$$
$$\leqslant C[\{M_n(Q; x_{kn}) + 1/|x_{kn}|\}|P_n^{(j+1)}(x_{kn})|$$

$$
\begin{aligned}
&+ \sum_{s=0}^{j-1} \{|x_{kn}|^{\langle s\rangle} n^2/a_n^{s+2+\langle s\rangle} + (n/a_n)^{s+1}/|x_{kn}|\}|P_n^{(j-s)}(x_{kn})|] \\
&\leqslant C|P_n'(x_{kn})|[\{M_n(Q;x_{kn}) + 1/|x_{kn}|\}\{M_n(Q;x_{kn}) + 1/|x_{kn}|\}^{1-\langle j+1\rangle} \\
&\times (n/a_n)^{j-1+\langle j+1\rangle} + \sum_{s=0}^{j-1} \{|x_{kn}|^{\langle s\rangle} n^2/a_n^{s+2+\langle s\rangle} + (n/a_n)^{s+1}/|x_{kn}|\} \\
&\times \{M_n(Q;x_{kn}) + 1/|x_{kn}|\}^{1-\langle j-s\rangle}(n/a_n)^{j-s-2+\langle j-s\rangle}] = \sum.
\end{aligned}
$$

Here if j is odd, then

$$
\begin{aligned}
\sum &\leqslant C|P'(x_{kn})|[\{M_n(Q;x_{kn}) + 1/|x_{kn}|\}(n/a_n)^j \\
&\quad + \sum_{s=0}^{j-1} \{n^2/a_n^{s+2} + (n/a_n)^{s+1}/|x_{kn}|\}(n/a_n)^{j-s-1}] \\
&\leqslant C|P_n'(x_{kn})|(n/a_n)^{j+1}.
\end{aligned}
$$

If j is even, then

$$
\begin{aligned}
\sum &\leqslant C|P'(x_{kn})|[\{M_n(Q;x_{kn}) + 1/|x_{kn}|\}(n/a_n)^j \\
&\quad + \begin{cases} \sum_{s;\text{even}} [\{n^2/a_n^{s+2} + (n/a_n^{s+1})/|x_{kn}|\}\{M_n(Q;x_{kn}) + 1/|x_{kn}|\} \\ \qquad (n/a_n)^{j-s-2}], \\ \sum_{s;\text{odd}} [|x_{kn}|(n^2/a_n^{s+3}) + (n/a_n^{s+1})/|x_{kn}|](n/a_n)^{j-s-1}] \end{cases} \\
&\leqslant C|P_n'(x_{kn})\{M_n(Q;x_{kn}) + 1/|x_{kn}|\}|(n/a_n)^j.
\end{aligned}
$$

For $x_{kn} = 0$ the theorem is shown by Theorem 3.5 and (3.15). □

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